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Quelques propriétés et applications du contrôle en temps minimal

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Introduction (Français)

Les problèmes de contrôle optimal sont à la croisée des chemins des systèmes dynamiques et de l'optimisation. Ces points de vue très différents ont tous deux de l'intérêt. D'un côté, on peut formuler le problème comme ceci : Comment peut-on trouver (ou prouver l'existence / unicité) des solutions d'une équation différentielle non autonome minimisant un certain coût, fixé à l'avance ? Selon l'autre point de vue, la question ressemblerait plutôt à : Comment résoudre un problème d'optimisation avec une contrainte dynamique ? Dans tous les cas ces problèmes sont de la forme suivante :

$$\begin{cases} \dot{x} = f(x,u), \ t \in [0, t_f] \\ x(0) = x_0, \ x(t_f) = x_f \\ C(u) = \int_0^{t_f} \varphi(x(t), u(t))dt \to \min . \end{cases} \tag{OC}$$

où f est une famille de champs de vecteurs paramétrés par $u \in U$, et U est un ensemble contraignant le contrôle, sur une variété M. Le choix, pour chaque temps t, d'un "meilleur" $u(t) \in U$, quand cela est possible, mène à un couple minimisant (x, u). Dans la seconde moitié du vingtième siècle, des techniques de nature géométrique ont été élaborées pour étudier ces problèmes. Nous rappellerons quelques notions mais ne ferons pas de tour complet de ces méthodes et nous renvoyons au livre [4] pour une présentation moderne. Elles reposent davantage sur l'aspect dynamique et sont plus proches des techniques employées dans cette discipline ainsi que de celles de la géométrie Riemannienne, par opposition aux méthodes venant du domaine de l'optimisation. La géométrie Riemannienne est de fait un cas particulier de géométrie sous Riemannienne, où la dynamique est linéaire en le contrôle, et la distribution de champs en question génère linéairement l'espace tangent en chaque point. Ces problèmes peuvent être formulés comme (OC), φ étant la norme au carré du contrôle.

Le coût est bien entendu d'une importance capitale, et le comportement des solutions de (OC) varie de manière drastique quand ce dernier change, en conservant la même dynamique. Cette thèse se concentre sur les problèmes en temps optimal, où le but est d'aller d'une position initiale x_0 à une position finale x_f en minimisant le temps d'arrivée. Ces problèmes sont étudiés dans le cadre où la dynamique initiale dépend du contrôle de manière affine. Ces systèmes, de la forme

$$\dot{x} = F_0(x) + \sum_{i=1}^{m} u_i F_i(x) \tag{\textit{Aff}}$$

restent très généraux : ils modélisent par exemple les système mécaniques et la plupart des problèmes de contrôle survenant dans la nature (modélisés par des EDO). Les systèmes mécaniques constituent l'application principale des travaux présentés ici, et seront notamment traités en détail certains système issus de la mécanique spatiale : le problème de transfert d'orbite, avec deux ou trois corps, au chapitre I, et IV.

Les méthodes géométriques sont tout indiquées pour l'étude des systèmes de contrôle affine. En effet, la plupart des propriétés de ces systèmes sont encodés dans les champs de vecteurs F_i supportant la dynamique, et dans leurs algèbres de Lie. La structure de ces algèbres

$$\mathbf{Lie}_x(F_0, F_1, \ldots, F_m), \quad x \in M$$

est primordiale, et de nombreux résultats classiques de contrôlabilité sont là pour en témoigner. L'intuition peut être donnée par un exemple célèbre : le problème consistant à garer sa voiture en créneau. Au vu des contraintes (dites, non holonomes), on ne peut se placer en face de la place en question et se déplacer perpendiculairement dans la direction voulue : aucun des champs de vecteurs ne permet ce mouvement. Il faut au contraire manœuvrer, et faire une série de mouvements pour accéder à ce déplacement : c'est une direction donnée par le crochet de Lie de ces champs, [37]. Cette structure reste très importante dans les problèmes de contrôle optimal. Tout au long de cette thèse, nous aurons besoin d'une hypothèse générique sur l'algèbre de Lie des champs, voir (A) au chapitre I, section 1.2. Le choix de minimiser le temps final est très particulier parmi tous les coûts disponibles, il est plus intimement lié à la dynamique initiale que les autres. En effet, selon le Principe du Maximum de Pontrjagin, les trajectoires optimales sont les projections des solutions d'un système Hamiltonien défini sur le cotangent de l'espace d'état par $H(x, p, p^0, u) = \langle p, f(x, u) \rangle + p^0 \varphi(x, u)$. Pour les problèmes en temps optimal, $\varphi = 1$, et sa présence dans le pseudo-Hamiltonien ne change rien aux courbes intégrales. On a alors affaire au relevé canonique au cotangent de la dynamique initiale. C'est encore plus marquant avec des systèmes de contrôle affine, où F_0, le *drift*, est le champ de la dynamique non contrôlée :

$$H(x,p,u) = H_0(x,p) + \sum_{i=1}^{m} u_i H_i(x,p). \tag{P-H}$$

Le hamiltonien est alors aussi affine en le contrôle et c'est une perturbation du relevé de la dynamique non contrôlée, si tant est qu'on considère des contrôles assez petits. Pour être plus précis, le Principe du Maximum affirme qu'il existe une courbe absolument continue $p(t)$ dans le fibré cotangent et une constante négative p^0, telles que H est maximum le long de $(x(t), p(t), u(t))$ parmi toutes les autres valeurs possibles pour le contrôle. Il est alors tentant de définir (quand il existe) $H^{\max}(x,p) = \max_{u \in U} H(x,p,u)$, et d'étudier les solutions du système $X_{H^{\max}}$. On appelle extrémales ces courbes (x,p), et leur projection x est une trajectoire extrémale. Quand le temps final n'est pas fixé, comme pour les problèmes en temps minimal, le P.M.P fournit une condition supplémentaire : $H \equiv 0$ le long des extrémales, ce qui nous donne encore $H = -p^0$ pour le temps optimal avec $H(x,p) = \langle p, f(x,u) \rangle$. Les extrémales dites *normales* correspondent au cas $p^0 \neq 0$, c'est le cas intuitif et général, tandis que quand $p^0 = 0$ on dit que l'extrémale est *anormale*. Quand on s'intéresse aux systèmes en temps minimal, les extrémales normales et anormales sont solutions de la même équation différentielle, mais les normales correspondent aux niveaux $H > 0$ et les anormales sont sur $H = 0$. L'étude des anormales est une des difficultés principales en contrôle optimal ; mis à part au chapitre III, nos résultats restent valides aussi bien dans le cas normal, qu'anormal.

Sans conditions supplémentaires sur le système (OC), $H^{\max}$ n'a aucune raison d'être régulier ou même bien défini. Le sujet principal de cette thèse est l'étude des singularités générées par cette condition de maximisation pour les systèmes en temps optimal. Nous nous intéresserons à plusieurs aspects et conséquences de ces singularités, selon différents points de vue. Une première conséquence, immédiate, est donnée par le comportement du flot extrémal au voisinage de ces singularités; nous verrons aux chapitres I et II qu'il peut y avoir bifurcation sur le lieu des points singuliers. La non-optimalité des extrémales, même locale, en est une autre. Nous verrons une condition permettant d'obtenir l'optimalité locale au chapitre III. Enfin, ces singularités peuvent aussi induire une absence d'intégrabilité, au sens de Liouville, pour le système Hamiltonien: le problème de Kepler, ainsi que son relevé au cotangent, sont des systèmes intégrables, pourtant nous montrerons au chapitre IV que son contrôle en temps minimal donne lieu à un système non intégrable.

Ces singularités coïncident avec les discontinuités du contrôle optimal (ou au moins, extrémal), appelé *switchings*. Eux mêmes se produisent lorsque prendre le maximum du pseudo-Hamiltonien sur les valeurs possibles du contrôle n'a pas de sens ou ne définit pas un u unique. L'Hamiltonien maximisé de (P-H) est

$$H^{\max}(z) = H_0(z) + \sqrt{\sum_{i=1}^{m} H_i^2(z)}, \; z \in T^*M \tag{Hmax}$$

(voir le calcul au chapitre I, section 1.1.). Les singularités se produisent clairement sur le lieu singulier

$$\Sigma = \{H_1 = \cdots = H_m = 0\}.$$

Le but de ce travail est donc la compréhension de ces singularités et de leurs conséquences, sous les différents angles mentionnés ci dessus. Une inspiration majeure a ainsi été l'article fondateur d'Ekeland [28]. Nous donnons maintenant un résumé des différents chapitres.

Chapitre I: Les singularités du contrôle en temps minimal des systèmes mécaniques

Le but de ce chapitre est l'étude du flot extrémal des systèmes de contrôles affine en temps minimal, dans un cas particulier des singularités possibles qui contient les systèmes mécaniques. Ce cas correspond à la zone Σ_- défini en 1.2.2. C'est à dire les systèmes de la forme

$$\ddot{q} + \nabla V(q) = u,$$

ou l'état q appartient à une variété de dimension 4 (ou $2n$ jusqu'à la section 1.2.3), V est une fonction lisse sur M (le potentiel) et u le contrôle : une force, que l'on choisit de façon à ce que l'état parte d'une position-vitesse initiale, et arrive à une position-vitesse finale, le plus rapidement possible. Ce problème ne serait pas bien posé sans borne sur le contrôle, et on le contraint ici à être dans une boule Euclidienne. De manière informelle, nous montrons que les singularités dans ce cas sont régulières, et que le flot est lisse par morceaux. Le chapitre débute par des rappels sur les notions de base et définitions de théorie du contrôle (contrôlabilité, ensemble accessible), puis nous introduisons les problèmes de contrôle optimal et les conditions nécessaires d'optimalité. En particulier, un énoncé précis est donné pour le Principe du Maximum de Pontrjagin avec temps final libre. Nous détaillons ensuite le calcul du hamiltonien *vrai*, au sens où il ne dépend plus du contrôle, dont les extrémales sont des relevés au cotangent des solutions optimales de (OC). Quelques exemples importants, qui seront repris tout au long de cette thèse, sont ensuite donnés.

Nous procédons ensuite en section 1.2.2 à une régularisation du champ de vecteurs donné par $X_{H^{\max}}$ d'après un éclatement polaire initié dans [2]. Ceci conduit à un résultat d'existence et d'unicité pour le flot extrémal, ainsi qu'à la construction d'une stratification sur laquelle le flot s'avère être lisse. La strate S_1 est celle de codimension 1, c'est aussi celle composée des conditions initiales menant au lieu singulier. Elle est construite comme la variété stable globale à une variété N, normalement hyperbolique pour le flot régularisé et contenue dans Σ. Ce caractère normalement hyperbolique ainsi que le fait que la dynamique soit triviale sur la variété N nous permet de conclure quand à la régularité des strates et du flot. Nous démontrons également la continuité du flot extrémal.

Dans la section suivante en 1.2.3, une étude de l'application de transition entre les strates est menée. Nous donnons une forme normale pour le flot, permettant de définir deux sections transverses aux strates. La forme normale est d'abord formelle, donnée par une série en les monômes résonants (ici, les monômes qui commutent avec la partie linéaire du flot), et l'on prouve une généralisation du théorème de Takens en ce sens. Il est alors possible de réaliser ces séries par des fonctions lisses, par le théorème de Malgrange (qui généralise celui de Borel, sur les séries réelles). Cette forme normale permet d'obtenir la régularité voulue, grâce à un éclatement projectif défini en fin de section. Nous obtenons des singularités en $d \ln d$ pour l'application de transition, où d représente la distance à la strate S_1.

Enfin, la dernière section de ce chapitre est consacrée à l'application aux systèmes mécaniques, puis va plus loin pour les problèmes de transfert d'orbite à deux ou trois corps. Ces problèmes ont une structure très particulière : la distribution de champs (F_1, F_2) commute. Ceci implique que les switchs sont en fait des rotations instantanées d'un angle π du contrôle. La fonction de switch, fonction qui s'annule aux temps de switchs, vérifie une équation différentielle linéaire d'ordre deux, et nous pouvons utiliser les théorèmes de comparaison à la Sturm pour contrôler l'intervalle de temps entre deux singularités. Dans le cas des problèmes de mécanique spatiale, nous donnons une borne supérieure sur le nombre de switchs pouvant arriver au cours d'un transfert en fonction de la distance aux collisions (les chocs avec les autres corps).

Chapitre II: Les singularités des systèmes affines en temps minimal

Le chapitre précédent a laissé ouverte la question des singularités se produisant en dehors de Σ_-, dans le reste du lieu singulier. Ces singularités ne peuvent pas se produire pour les systèmes provenant de la seconde loi de Newton, comme il est montré en section 1.2.4. Nous répondons néanmoins à cette question de manière complète pour les systèmes de contrôle à double entrée sur les variétés de dimension 4 (avec un contrôle contenu dans une boule Euclidienne). Les strates stable et instable construites au chapitre I fusionnent en un point d'équilibre nilpotent dans Σ pour le système régularisé. L'analyse des switchs et du flot extrémal en ce point requiert ainsi d'autres outils. Nous commençons, en section 2.2, par unifier les points de vue en donnant un système à paramètre α, contenant notre problème. A la bifurcation $\alpha = 0$, dans (2.2.3), se produit la rencontre des variétés stable et instable.

Nous donnons ensuite une description du flot singulier - flot contenu dans Σ - comme le flot d'un certain Hamiltonien lisse, et une comparaison est faite avec le cas, plus simple, du système mono-entrée, quand le contrôle est scalaire. Nous discutons aussi du même système avec une contrainte différente sur le contrôle : quand $U = [-1, 1]^m$. De nombreuses études ont été menées dans ce contexte, et nos singularités peuvent être apparentées à des switchs doubles, mais on ne peut, en pratique, les traiter de la même manière que dans [48]. Une extrémale dite *bang* est en essence une extrémale qui reste en dehors du lieu singulier, et son contrôle associé est lisse. Une concaténation d'extrémales bang est appelée bang-bang, c'est le cas des extrémales de la strate S_1 sur un temps plus grand que le temps de switch. En dehors de la zone Σ_- un flot existe dans le lieu singulier, on dit que les extrémales correspondantes sont singulières, et il peut exister des connexions bang-singulière.

Nous présentons d'abord le cas le plus simple, celui ou les extrémales n'entrent pas en contact avec le lieu singulier ; cette zone est notée Σ_+, et une simple estimation à la Gronwall suffit. En section 2.2.3, nous traitons la bifurcation, et la fusion des variétés stable et instable. Nous procédons à un éclatement quasi-homogène pour étudier l'équilibre nilpotent du champs régularisé. Cet éclatement, introduit par Dumortier dans [27], consiste à trouver les bons poids pour chaque coordonnée, de façon à ce que l'anisotropie désingularise le système, ce qu'un seul éclatement en coordonnées sphériques classiques n'aurait pu effectuer. Une fois l'équilibre désingularisé, nous procédons à l'étude du système éclaté sur la (demi-)sphère, et obtenons cinq nouveaux équilibres, tous hyperboliques ou semi-hyperboliques. Cette étude est grandement facilitée par la dimension deux : nous pouvons utiliser la panoplie de résultats existants dans le plan. Le théorème de Poincaré-Bendixson nous permet de relier les variétés instables de ces équilibres avec les variétés instables des autres, si l'on prouve l'absence de trajectoires périodiques. Nous parvenons à le démontrer en utilisant la formule de Stockes sur un domaine bien choisi de l'hémisphère transverse au champ. Nous exhibons, dans la moitié des cas une direction stable provenant de l'extérieur de la sphère, prouvant au passage le théorème 2.2 sur l'existence de switchs dans ce cas. Deux possibilités se présentent alors : selon le signe du crochet de Poisson des relevés des deux champs H_{12}, le contrôle associé à l'extrémale peut être continu ou présenter une π-singularité. Une analyse similaire à celle du chapitre I permet alors d'obtenir également une stratification sur laquelle le flot est lisse. Avant de clore ce chapitre en fournissant un exemple de système de contrôle affine présentant ces propriétés, nous faisons une remarque informelle sur le fait suivant : au vu du portrait de phase donné, toute la situation est contenue dans le cas nilpotent.

Chapitre III : Conditions suffisantes d'optimalité pour les systèmes affines en temps minimal

L'existence et l'unicité du flot donné de l'Hamiltonien maximisé donné par le Principe du Maximum est d'une importance capitale : s'il existe une trajectoire optimale, c'est alors la projection de l'extrémale en question. Il existe des résultats sur l'existence globale d'une trajectoire optimale, comme le théorème de Filippov, [24], mais la plupart du temps il est impossible d'en satisfaire les hypothèses sans en faire d'autres, très fortes, sur le système de contrôle initial. Pour les problèmes de transfert d'orbite par exemple, on peut appliquer le théorème de Filippov si on suppose que le transfert s'effectue dans un domaine compact bien choisi, voir [23].

En général, une approche plus raisonnable est le point de vue local, et la question de l'optimalité devient : jusqu'à quel temps (ou point), une trajectoire extrémale reste-t-elle optimale parmi toutes les trajectoires admissibles proches. Ce temps est appelé temps conjugué et le point associé, point conjugué. Ces concepts sont définis comme les moments de verticalité pour les solutions du système linéarisé : c'est ainsi qu'on a besoin d'une grande régularité, au minimum $\mathcal{C}^2$, pour le hamiltonien maximisé. Nous rappelons la théorie classique brièvement en section 3.1, et énonçons un théorème d'optimalité locale dans le cas lisse.

Dans notre contexte, $H^{\max}$ n'est que continu, et son flot est lisse sur une stratification. Nous dressons une comparaison entre notre situation et celle où le contrôle est contenu dans un polyèdre, où les résultats nécessitent une hypothèse sur une variation du second ordre obtenue en considérant un sous-système où les temps de switchs varient. En section 3.2, nous énonçons notre condition d'optimalité locale. Le reste du chapitre est dédié à sa preuve, par des méthodes issues de la géométrie symplectique. L'idée est la suivante : si toutes les courbes au voisinage de notre trajectoire extrémale de référence peuvent être relevées de manière unique au cotangent, au vu de la définition du pseudo-Hamiltonien, il est alors naturel d'utiliser la forme de Poincaré-Cartan $pdx - Hdt$ pour comparer leurs coûts avec celui de la trajectoire de référence. Dans le cas du temps minimal, on peut utiliser la forme de Liouville $\lambda = pdx$. La première étape consiste à construire une perturbation Lagrangienne $\mathcal{L}$ de $T^*_{x_0}M$, qui intersecte transversalement la strate S_1. Bien choisie, la projection canonique $\pi : T^*M \to M$ est une bijection du graphe par le flot de $\mathcal{L} \cap S_1$ sur son image, en faisant attention à recoller correctement les morceaux se situant avant, et après le temps de switch. Les courbes admissibles dans un voisinage ont alors un unique relevé, et il faut alors comparer leur coûts. Nous concluons utilisant le fait que la forme de Liouville est exacte, grâce au caractère Lagrangien de $\mathcal{L}$. Un résultat analogue avec des conditions très différentes ne s'appliquant pas ici à été donné dans [3]. Enfin, la section 3.4 de ce chapitre est consacrée à une preuve de la régularité de la fonction valeur au voisinage d'un point final où l'extrémale est optimale. Nous obtenons la même régularité lisse par morceaux que pour le flot.

Chapitre IV : Non-intégrabilité du problème de Kepler en temps optimal

Ce dernier chapitre a pour but l'étude des effets des singularités du contrôle en temps optimal sur l'intégrabilité (au sens de Liouville) dans le problème de Kepler : un corps soumis à l'attraction gravitationnelle d'un centre fixe. Nous commençons par de brefs rappels sur les systèmes Hamiltoniens et leur intégrabilité, et la théorie de Galois différentielle pour les équations linéaires. Le problème de Kepler est intégrable, et en fait même super-intégrable, et le théorème d'Arnold-Liouville s'applique : l'espace des phases est feuilleté en tores stables par la dynamique, sur lesquels, dans les bonnes coordonnées, le mouvement est quasi-périodique (les trajectoires sont des droites sur le relevé universel). Le but de ce chapitre est de montrer que cette structure très régulière est détruite par le contrôle de ce problème en temps minimal.

L'Hamiltonien du problème de Kepler en temps minimal est donné par le maximum sur toutes les valeurs possibles du contrôle, du hamiltonien du problème de Kepler relevé au cotangent H_0 - qui reste intégrable - auquel on ajoute la composante venant du contrôle :

$$H = H_0 + u_1 H_1 + u_2 H_2,$$

où les H_i sont les relevés canoniques au cotangent des champs F_i.

Les applications de la théorie de Galois différentielle sont rares dans le contexte du contrôle optimal, et nous donnons une preuve de non intégrabilité dans la classe des fonctions méromorphes. Avant d'énoncer le théorème de Moralès et Ramis dont nous nous servirons, nous donnons une explication accompagnée du Lemme de Ziglin : les intégrales premières d'un système Hamiltonien génèrent des intégrales premières de son linéarisé - ce fait avait été réalisé par Poincaré : ceci fait du groupe de Galois différentiel de l'équation linéarisée un outil d'une importance capitale. Il est défini comme le groupe des automorphismes différentiels qui préserve le corps de base, aussi, ses éléments préservent les relations entre les solutions. Ce sont des groupes de Lie (et des groupes algébriques), et le théorème de Moralès-Ramis indique que, si le système Hamiltonien est intégrable, la composante connexe de l'identité de ce groupe doit être Abélienne. Une autre caractérisation de l'intégrabilité des systèmes Hamiltoniens a été découverte plus tôt par Ziglin, dans [60], via l'étude du groupe de monodromie. C'est le groupe de matrices obtenus par l'action du groupe fondamental sur les solutions du système, par prolongement analytique. Il est contenu dans le groupe de Galois G de l'équation variationnelle, et est en fait dense dans ce dernier. Ceci va nous permettre d'améliorer notre résultat : de la classe des fonctions rationnelles, la non-intégrabilité se transmettra à celle des fonctions méromorphes.

La première étape consistera à trouver une solution de notre problème de Kepler en temps minimal, c'est l'une des difficultés de la théorie. En général, ceci est accompli en utilisant une sous-variété stable sur laquelle le système est intégrable. Il est alors difficile de prouver, via le théorème de Moralès-Ramis, la non-intégrabilité d'un système se situant "très loin" de l'intégrabilité, qui ne possèdera pas de tel sous-espace. Nous donnons une sous-variété stable sur laquelle toutes les trajectoires sont des trajectoires de collisions. Il existe une intégrale première supplémentaire sur cette sous variété, c'est celle bien connue du problème de Kepler auquel une force constante a été ajoutée. Après avoir exhibé une trajectoire de collision particulière, nous étudions l'équation variationnelle réduite, et les variations normales à notre sous espace stable sont considérées. En extrayant une équation scalaire dont le groupe de Galois est contenu dans G, on montre que celui-ci contient le groupe de Galois d'une équation hypergéométrique. Les tables de Kimura [36], au vu de nos paramètres, nous permettent de conclure.

Introduction

Optimal control problems are at the interaction of dynamical systems and optimization. The two very different viewpoints are of importance: on the one hand, it can be formulated as follows: How can we find a solution - or prove its existence - to a non-autonomous differential equation which minimizes a certain cost? On the other hand, from an optimization point of view, the question would rather take the form: How can we minimize a certain functional, while having a dynamical constraint? In any case, these problems, when an integral cost is considered, can be written as:

$$\begin{cases} \dot{x} = f(x,u), \ t \in [0, t_f] \\ x(0) = x_0, \ x(t_f) = x_f \\ C(u) = \int_0^{t_f} \varphi(x(t), u(t)) dt \to \min, \end{cases} \tag{OC}$$

where f is a family of vector fields on a manifold M, parametrized by the control $u \in U$, and U is a set constraining the control. The choice, for every time t, of a best $u(t)$, such that the pair (x, u) minimize the cost gives a solution of the optimal control problem (OC). In the second half of the last century, geometric approaches were elaborated in order to better understand and treat those problems. We recommend the book [4], to the reader interested in a modern and complete presentation of these methods, as we will not give a complete overview here. They rely more on the dynamical aspect of optimal control problems, and the techniques used are closer to the one of dynamical systems and Riemannian geometry. This last field happens to be a sub-case of sub-Riemannian geometry which is itself a particular case of optimal control system, where one will attempt to minimize the L^2 norm of the control.

Depending on the cost, from the same initial control system, one ends up with very different behaviors regarding optimal solutions. This thesis tackles the issue of time optimality (i.e., going from a initial point, to a final one, " as fast as possible "), in problems where the dynamics has an affine dependence in the control. Those control-affine systems

$$\dot{x} = F_0(x) + \sum_{i=1}^{m} u_i F_i(x) \tag{Aff}$$

are very general in the sense that they model every mechanical system, and most of the control problems found in nature (modeled by ODE's). Mechanical systems will be our main focus of application, and we will go further into details with issues from space mechanics, as orbit transfer or rendez-vous problems, with two or three bodies. Control-affine systems are well-suited for the use of geometrical methods, indeed, most of the properties of such systems are encoded in the vector fields F_i, and their Lie algebras. The structure of the Lie algebras

$$\mathbf{Lie}_x(F_0, F_1, \ldots, F_m), \quad x \in M$$

is primordial, numerous theorems in controllability theory are here to prove it. The famous example of parking a car in a slot gives the intuition: with the constraints involved (so called non-holonomic) one cannot make the move directly to the slot, it would be moving along a direction which is not given by any vector field of the system, but has to do a series of moves that actually correspond to the direction of the Lie brackets of some of the vector fields, [37]. This structure remains of tremendous importance in optimal control problems. Along this work, to obtain our results, we will often make generic assumptions on the Lie brackets of the vector fields involved, see, for instance chapter I, assumption (A). The minimization of the final time is a peculiar choice of cost, it is intimately linked with the initial dynamics. Indeed, from Pontrjagin Maximum's Principle, a control u is optimal if its associated trajectory is the projection of the solution of the Hamiltonian system given by $H(x, p, p^0, u) = \langle p, f(x, u)\rangle + p^0\varphi(x, u)$. For optimal time problems, $\varphi = 1$ and the integral curves, are then just solution of $H(x, p, u) = \langle p, f(x, u)\rangle$, the lifted dynamics. This fact is even more relevant with affine control systems, where the *drift* F_0 is the uncontrolled initial dynamics. The Hamiltonian

$$H(x, p, u) = H_0(x, p) + \sum_{i=1}^{m} u_i H_i(x, p) \tag{P-H}$$

is affine in the control as well, and, provided the controls are in a small ball of radium ε for instance, the Hamiltonian vector field is a small perturbation of the lift of the initial dynamics. More precisely, the Maximum's Principle state that there exists a curve $p(t)$ in the cotangent bundle of the initial phase space, and a constant p^0 such that H is maximum along this curve among every value in the control set. One would be then tempted to invest his time in studying solutions of $H^{\max}(x,p) = \max_{u \in U} H(x,p,u)$. Those are called extremals, and their projection on the phase space, extremal trajectories. For minimum time systems,

$$\begin{cases} \dot{x} = f(x,u),\ t \in [0, t_f] \\ x(0) = x_0,\ x(t_f) = x_f \\ t_f \to \min \end{cases} \tag{\textit{Tmin}}$$

the final time in (*Tmin*) is let free and another condition arises from the P.M.P.: $H = -p^0$ along every extremal. When p^0 is non-zero, the extremal is called normal, otherwise, we call it abnormal. Thus, abnormal extremals are the ones such that $H = 0$. Abnormal extremal are an issue of importance in optimal control, and often one of the main difficulty, however, except for chapter III, our results are valid both in the normal and abnormal case. Obviously, without further assumptions on the control system, and due to the maximization condition, the Hamiltonian vector field of the maximum of H over every control values has no reason to be well defined. Throughout this work, we will be interested in the singularities generated by this maximization condition for minimum time control systems. In a way, this thesis focuses on several aspects and consequences of this singularities. The irregularities and bifurcations in the local behavior of the dynamics are one of these consequences. The lack of optimality (at least, locally) of certain extremal trajectories is another one. Finally, non-integrability of the minimum time Kepler problem is also result from those singularities: The Kepler problem is integrable, and so is its lift to the cotangent bundle. Hence, the singularities generated by the optimization problem must be responsible for the lack of integrability. Those singularities coincide with discontinuities of the optimal (or at least, extremal) control, called switchings. Themselves occur when the maximum among all control values does not define a unique u. The maximized Hamiltonian of (P-H) is

$$H^{\max}(z) = H_0(z) + \sqrt{\sum_{i=1}^{m} H_i^2(z)},\ z \in T^*M \tag{Hmax}$$

see chapter I section 1.1. Clearly, the singularities are occurring at

$$\Sigma = \{H_1 = \cdots = H_m = 0\}.$$

One of the main topics of this thesis is to understand those singularities from different points of views. In that regard, this work was deeply inspired by the pioneering paper of Ekeland, [28]. We now give a summary of the work undertaken in this manuscript.

Chapter I: Singularities of minimum time control for mechanical systems

We study the extremal flow of minimum time control-affine systems, through a particular case meant to be applied to the control of mechanical systems:

$$\ddot{q} + \nabla V(q) = u,$$

where the state q lives in some manifold M, V is a smooth function on M (the potential) and u is the control: A well chosen force added to the motion, in order to minimize the time of arrival to a final state. This case corresponds to the zone Σ_- defined in 1.2.2. Informally, we prove in this chapter that these problems have regular singularities, and their flow is piecewise smooth. We begin by recalling basics and more advanced notions of modern optimal control. We recall basic definitions, controllability notions and the necessary conditions for optimality for optimal control systems with free final time, giving a precise statement for Pontrjagin's Maximum Principle. We then give a computation of the maximal Hamiltonian, whose extremals are lifts of optimal time solutions of (*Tmin*). The remainders end by a few examples from space mechanics, namely the controlled Kepler and restricted three body problems, that will be of use throughout this thesis. We restrict our study to a part of the singular locus which is meant to contain mechanical systems, namely, the set Σ_- defined in section 1.2.1. Inspired by [2], we provide a regularization of the dynamics given by (Hmax). Through this, we manage to give an existence and uniqueness result as well as a stratification of a neighborhood O of the singular locus

$$O = S_0 \cup S_1 \cup \Sigma$$

on which the extremal flow is smooth in section 1.2.2. The stratum S_1 of codimension one is actually the global stable manifold to a normally hyperbolic submanifold for the regularized vector field contained in Σ. The stratum of codimension two is the singular set Σ_-. Indeed, the regularized dynamics, in this region is actually normally hyperbolic to a submanifold of N in Σ_-, and on N the dynamics is trivial: This allows us to conclude regarding the regularity of the strata and of the flow. Furthermore, we prove the continuity of this flow.

Then, in 1.2.3 we study the transition from a stratum to another one, giving the precise type of singularity occurring: The associated Poincaré mapping turns out to be in the log-exp category. To be able to compute the transition, we use a normal form for our regularized system. We begin by computing formally the normal form: We generalize a theorem of Takens to prove that it is given by a power series in resonant monomials (ie, monomials commuting with the linear part of the vector field). Then, one can use a generalization of the celebrated theorem from Borel to realize this formal series by a smooth function. Once the normal form is obtained, the desired regularity for the transition is achieved by a particular type of blow up. Singularities are of type "$d \ln d$", where d here represents the distance to the stable stratum S_1 previously mentioned.

The last section of this chapter is devoted to applications to mechanical systems, and more precisely to the orbit transfer problem. The goal is to make a transfer from an initial orbit to a final one as fast as possible with a bounded control. These problems have a special structure: The distribution (F_1, F_2) actually commutes. This implies a special type of switchings called π-singularities: Instant rotations of angle π of the control. The switching function actually verifies an order two linear differential equation, and using a comparison "à la Sturm", we control the number of switchings occurring during a transfer by giving an upper bound linked with the distance to collisions.

Chapter II: Singularities of minimum time control-affine systems

The last chapter left open the question of the behavior and regularity of the extremal flow around $\Sigma \backslash \Sigma_-$. This singularities cannot happen for systems coming from Newton's 2nd Law, as shown in section 1.2.4. We provide a complete picture of the behavior of the minimum time extremal flow for generic double input control-affine systems, when the control evolves in a ball. More precisely, the stable and unstable strata of the previous chapter end up merging, at a nilpotent equilibrium in Σ for the regularized system. Hence, the analysis of the switchings at this point required higher order tools. In section 2.2, we have made a effort to give a reunification of the three types of singularities occurring in Σ: A family of dynamical systems with a parameter is given, the bifurcation between the three cases occurring when the parameter α in (2.2.3) crosses zero.

We begin by studying the singular flow - the flow that lies in the singular locus, and dressing a comparison with the single input case. We also have a discussion on the differences between our configuration and the case where the control set is polyhedral. Our singularities can *à priori* be compared to double switchings in their cases, but it cannot be treated the same way as in [48], for instance. Basically, a *bang* extremal remains outside the singular locus, and its associated control is smooth. A concatenation of bang extremal is called bang-bang. In chapter I, extremals were either bang or bang-bang, though singular extremals - extremals lying inside Σ - exist in the other cases. The singular flow is smooth, and this chapter also answers the question of the existence of bang-singular connexions. We first deal with the easiest case: Σ_+, where the regularized system presents no equilibrium in Σ, implying no switching occurs whatsoever, this can be proved through a simple exponential estimation. Then we tackle the problem of the merging stable and unstable manifold and what happens at the bifurcation. The nilpotent equilibrium is studied through a quasi-homogeneous blow up introduced by Dumortier twenty five years ago in [27]. Successive usual (spherical) blown-ups can be applied to this kind of problems, but there is no guarantee that the desingularization will be successful, it is highly more technical and a difficulty is to go back to the original problem. By finding the right weight when blowing up, most of the time applying this process one time is enough. Once the desingularization achieved, the blown up point turns into a 2-sphere with five new equilibria. The process was successful since they all turn out to be hyperbolic or semi hyperbolic. We take advantage of working in dimension two: There exists a great amount of powerful results for dynamical systems in the plane. We use freely the Poincaré-Bendixson theorem to connect the stable and unstable manifolds from every equilibria in the hemisphere. To that end, one must exclude the existence of periodic trajectory, which we do by considering a well chosen domain, transverse to the vector field and use Stokes theorem to conclude. The existence of a stable direction coming from outside the sphere proves, via an estimation on the initial time, that there exists an extremal going to the singular locus in the nilpotent case (Σ_0, below). Regarding the switching on the associated extremal control, there are two alternatives depending on the sign of the Poisson brackets of the lift of F_1 and F_2, H_{12}. The control can be continuous or presents a π-singularity. Then, a similar analysis of the one in chapter I can be led to build a stratification on which the flow is smooth too. Finally, an informal commentary is made: Through the phase portrait, one can see that the whole situation is contained in this nilpotent case. We end this chapter by providing an example of a non-mechanical control-affine system with the kind of trajectory exhibited in theorem 2.2.

Chapter III: Sufficient conditions for optimality of minimum time control-affine systems

Existence and uniqueness for the flow given by the Maximum Principle is of importance in optimal control: Provided an optimal trajectory exists, it has to be the extremal. Even though global existence theorems exist, as for instance, Filippov's [24], it is most of the time difficult to satisfy its their hypothesis without very strong assumptions on the control system. For orbit transfers for instance, it can be applied provided the transfer remains in a compact domain, see [23].

In general, a more reasonable approach is the local viewpoint. The question becomes: Until what time, or point, does a trajectory remain optimal among all nearby admissible trajectories (by an admissible trajectory we mean a solution of the control system for some control $u \in L^\infty([0, t_f])$). This particular time is called a conjugate time, and the associated point is a conjugate point. These concepts are defined by verticality moments of the solutions of the variational equation. That means their definition requires at least a $\mathcal{C}^2$ regularity for the Hamiltonian. In section 3.1, we recall the classical smooth theory of conjugate points, and a local optimality theorem for smooth optimal control systems.

In our case, $H^{\max}$ is only continuous, and its flow is regular on a stratification. We make a comparison with the situation obtained when the control lies in a polyhedron and the results obtained in that case, using, in general, a second order variation for a finite dimensional sub-system. In the next section, we state our result on local optimality among all nearby admissible $\mathcal{C}^0$-curves for extremal trajectories satisfying a disconjugacy condition on the smooth stratum. This condition can easily be checked numerically. We prove this result in section 3.3, using symplectic methods. The idea is the following: If every curves in the neighborhood of the reference extremal trajectory can be uniquely lifted to the cotangent bundle, then one can use the Poincaré-Cartan form $pdx - Hdt$ to compare their cost with the reference one. Indeed, it is the natural tool from the definition of the pseudo-Hamiltonian. In the minimum time case, one can use the Liouville form $\lambda = pdx$. We start by building a Lagrangian manifold $\mathcal{L}$, that is a perturbation of $T^*_{\bar{x}_0}M$, so that the intersection with the strata S_1 is a smooth submanifold. The canonical projection

$$\pi : T^*M \to M$$

is one to one on the image of the graph submanifold by the flow, ie, on

$$\{(t, e^{tH^{\max}}(\mathcal{L} \cap S_1)), \quad t \in [0, t_f] \backslash \{\bar{t}\}\}$$

where $\bar{t}$ is the switching time. We can then lift all the nearby admissible curves. The last step to compare their final time consists in proving the Liouville form is exact, this conducts to the local optimality result.

The last section, 3.4, of this chapter is devoted to the proof of the regularity of the field of extremals. This implies the regularity of a upper bound to the value function: we obtain a minimum time depending piecewise smoothly of the end point under assumptions similar to the ones in theorem 3.2.

Chapter IV: Non-integrability of the minimum time Kepler problem

This last chapter is dedicated to study the effect of minimum time singularities on the Liouville integrability in the context of the Kepler problem: One body submitted to the gravitational attraction of a fixed center of mass. The chapter opens on a brief review on Hamiltonian systems, integrability and Galois theory for linear differential equations. The Kepler problem is integrable, and as such, the Liouville-Arnold theorem applies and the phase space is foliated in stable tori, on which the motion is quasi-periodic: The angular coordinates on each torus evolves linearly with time. The aim of this chapter is to prove this structure is destroyed by controlling this problem while minimizing the final time. The Hamiltonian of the minimum time Kepler problem is given by:

$$H^{\max} = \max_u \{H_0 + u_1 H_1 + u_2 H_2\}$$

where is H_i being the canonical lift of the fields F_i and H_0 is the lifted Kepler dynamics, which remains integrable.

Applications of the Galois differential theory methods are rare in the context of optimal control. We give a proof of non-integrability here in the class of meromorphic functions. We describe Ziglin's Lemma, but Poincaré already noticed that fact: The first integrals of an Hamiltonian system generate first integrals of its linear part. This translates, through the celebrated Moralès-Ramis theorem, in terms of the Galois group of the variational equation. It is, by analogy with the classical theory, defined as the group of differential automorphisms which preserve the base field. These are algebraic Lie groups and when the system is integrable, these groups are virtually Abelian: The connected component of the identity has to be Abelian. Another characterization of integrability have been given earlier by Ziglin, through the study of the monodromy group. It is obtained by the action of Π_1, the fundamental group, on the solutions by analytic continuation, and is included in the differential Galois group G of the linearized equation. It is actually dense in G for the Zariski topology. This fact will allow us to upgrade our non-integrability result from the class or rational functions to the class of meromorphic ones.

The first step, and it is one of the difficulties this theory, is to find a solution of the initial (non-linear) differential equation. Most of the time, this is achieved by considering a stable submanifold on which the dynamics is simpler, and often, even integrable. This constitutes a weakness of the theory, according to Ramis, because a system which is very far from integrability does not possess this kind of object. We find a stable surface and an independent first integral on it, which turns out to be a first integral of the Kepler motion with a constant force added. With the choice of our stable surface, our particular solution is the lift of a collision trajectory. We then give a computation of the reduced variational equation is given, keeping only the normal variations to our stable surface. We eventually write a scalar equation from the vectorial one, whose Galois group is contained in G. The solutions have the form of a product in which one term is a primitive of the hypergeometric Gaussian function. The Galois group of such functions is thus included in G, and they were classified by Kimura in his paper [36]. Given our parameters, it implies that the Galois group is not even solvable.

Notations.

- Smooth will mean of class $\mathcal{C}^\infty$.
- M is a 4-dimensional smooth manifold, $\mathcal{C}^k(M)$ the set of smooth real valued functions on M
- Let F_0, F_1 and F_2 be smooth vector fields on M.
- Let TM and T^*M be respectively the tangent and cotangent bundles of M, and $\pi : T^*M \to M$ be the canonical projection.
- Let $|v|$ be the standard Euclidean norm of a vector $v \in \mathbb{R}^n$.
- Let $[\cdot,\cdot]$ be the standard Lie bracket of vector fields on manifolds. For two vector fields F_i, F_j we denote their Lie bracket by $F_{ij} = [F_i, F_j]$.
- For a function $H : T^*M \to \mathbb{R}$, we denote X_H the associated Hamiltonian vector field.
- Similarly, let $\{\cdot,\cdot\}$ be the standard Poisson bracket on T^*M: In Darboux coordinates (x,p), $\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial p}$. For two smooth Hamiltonians H_i on T^*M, we denote their Poisson bracket by $H_{ij} := \{H_i, H_j\}$, so that if z is a Hamiltonian trajectory of H, $\frac{d}{dt}f(z(t)) = \{H, f\}(z(t))$. More generally, $H_{i,i_1\ldots,i_n} := \{H_i, H_{i_1\ldots i_n}\}$.
- For $f \in \mathcal{C}^\infty(T^*M)$, let $\operatorname{ad} f : \mathcal{C}^\infty(T^*M) \to \mathcal{C}^\infty(T^*M)$, with $\operatorname{ad} f(g) = \{f, g\}$.
- For an normally hyperbolic equilibrium point $z \in M$, let $W^s(z)$ (respectively $W^u(z)$) be its stable (respectively unstable) manifold.
- Let $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ be the standard two-dimensional symplectic matrix.

Chapter 1

Singularities of minimum time control for mechanical systems

Abstract

We prove the extremal flow of minimum time mechanical systems is piecewise smooth, and smooth on a stratification. There is at most one switching in a neighborhood of the singular locus. We give the type of singularities of the transition map between the strata by proving it belongs to the log-exp category. The minimum time two and restricted three body problems are studied as an application, and we give a upper bound on the number of switchings in that case.

1.1 Quick introduction to geometric control & necessary conditions for optimality

In this first section we introduce some of the important notions, tools and main classical result of geometric control theory, that will be of use throughout this thesis.

1.1.1 Controllability

Let M be a n-dimensional smooth connected manifold (the state space), and U be an arbitrary set of $\mathbb{R}^m$ (the control space). A control system is given by a family of vector fields $f : M \times U \to TM$. This gives us a family of dynamical systems

$$\dot{x} = f(x, u) \tag{1.1.1}$$

If one allows the parameter u to change at each time t, we get a classical control problem. Under suitable hypotheses, namely, smoothness of f for instance, by Carathéodory's theorem, for any control $u \in \mathcal{U}$, system (1.1.1) has a unique solution. For a control $u \in \mathcal{U} := L^\infty([0, t_f], U)$, and $x_0 \in M$ denote $x(t, x_0, u)$ the solution of (1.1.1) with $x(0, x_0, u) = x_0$, or just $x(t)$ when there is no ambiguity. We call $\mathcal{U}$ the set of admissible controls. Define the endpoint mapping

$$E_{x_0} : u \in L^\infty([0, t_f], U) \mapsto x(t_f, x_0, u) \in M.$$

There are several notions of controllability for a control system. Let us first define the attainable set.

Definition 1.1 (Attainable set)
Let $x_0 \in M$, $t \in \mathbb{R}$, the attainable set from x_0 at a time t is defined by $\mathcal{A}_{x_0}(t) := \{x(t, x_0, u), u \in \mathcal{U}\}$. The attainable set from x_0 is $\mathcal{A}_{x_0} = \cup_{t \in \mathbb{R}_+} \mathcal{A}_{x_0}(t)$.

We say a control system is controllable from x_0 if $\mathcal{A}_{x_0} = M$. So the system is controllable from x_0 if the endpoint map E_{x_0} is onto. It is controllable if this holds for any $x_0 \in M$. We will be interested mainly in control-affine systems, meaning, systems of the form

$$\dot{x} = F_0(x) + \sum_{i=1}^{m} u_i F_i(x), \; u(t) \in U \tag{\textit{Aff}}$$

For instance, every controlled mechanical system (see 1.1.4 below) is modeled by affine control systems. They can also be seen as a generalization of sub-Riemannian systems (linear on the control), with a drift added. On a sub-Riemannian system $\dot{x} = \sum_{i=1}^{m} u_i F_i(x)$, the state cannot only have tangent directions along the F_i's but also their Lie brackets (think of the classical example of parking a car): The bracket generating condition

$$\mathbf{Lie}(F_1, \ldots, F_m)(x) = T_x M$$

for all $x \in M$ is sufficient to ensure controllability under suitable hypothesis on the control set U (namely, that its convex hull contains a small ball around $0 \in \mathbb{R}^m$). This result is known as the Chow-Rashevskii theorem. One can intuitively understand that it is not the case in control-affine systems since the drift can be too strong with respect to the control, and thus forbid controllability. There has been a great amount of results on controllability for affine control systems, see for instance Sussman's conditions or [35]. When the drift is reasonable, ie, does not take the system to infinity, one can imagine that controllability can be achieved. We recall a precise result below, a vector field is called *recurrent* if its flow has a dense subset of recurrent points. In the systems we are interested in (namely systems from space mechanics), the drift has a chance to be recurrent, and we have:

Theorem 1.1
Consider an control-affine system (Aff). *If*

(i) F_0 *is recurrent*

(ii) the convex hull fo U *contains a neighborhood of 0.*

(iii) $\mathbf{Lie}_x(F_0, F_1, \ldots, F_m) = T_x M$ *for all* $x \in M$

then, the system is controllable.

This is a consequence of the orbit theorem, [35]. The bracket generating property is fundamental, and in what follows we will use the following notation for the Lie brackets involved in control-affine problems: $[F_i, F_j] := F_{ij}$, and iterating $[F_k, F_{ij}] = F_{kij}$ and so on. The same notation will be used for the Poisson brackets: $\{H_i, H_j\} := H_{ij}$.

1.1.2 Geometric optimal control

We would like to control our dynamical system minimizing a criterion, along the trajectories. This criterion will be an integral cost, leading to the following problem:

$$\begin{cases} \dot{x} = f(x,u),\ u(t) \in U,\ t \in [0,t_f], \\ x(0) = x_0,\ x(t_f) = x_f, \\ C(u) = \int_0^{t_f} \varphi(x(t),u(t))dt \to \min . \end{cases} \tag{1.1.2}$$

where $\varphi : M \times U \to \mathbb{R}$ is a smooth (or continuous) function. In order to understand the mechanisms of the classical necessary conditions for optimality, let us introduce the extended system

$$\begin{cases} \dot{x} = f(x,u),\ u(t) \in U,\ t \in [0,t_f], \\ \dot{y} = \varphi(x,u), \\ x(0) = x_0,\ x(t_f) = x_f, \\ y(0) = 0,\ y(t_f) = C(u). \end{cases} \tag{1.1.3}$$

and the extended end-point map

$$\tilde{E}_{x_0} : u \in L^\infty([0,t_f],U) \mapsto (x(t_f,x_0,u), y(t_f,x_0,u)) \in M.$$

If a control u is optimal, then $\tilde{E}_{x_0}(u) \in \partial\tilde{\mathcal{A}}_{(x_0,0)}$ (with obvious notation) and if the control set U is open, one can find Lagrange multipliers $(P,P^0) \in \mathbb{R}^n \times \mathbb{R}$ such that $P.dE_{x_0}(u) = -P^0.dC(u)$. This couple of Lagrange multipliers are linked with the adjoint state in the Hamiltonian formalism given by Pontrjagin to optimal control problems in the late 60's.

Definition 1.2 (Pseudo-Hamiltonian)
The pseudo-Hamiltonian of system (1.1.2) is defined by

$$H(x,p,p^0,u) = \langle p, f(x,u)\rangle + p^0\varphi(x,u),$$

$(x,p) \in T^*M,\ u \in U,\ p^0 \in \mathbb{R}.$

Being in the boundary of the attainable has the following consequence:

Proposition 1.1
If u is an admissible control such that $E_{x_0}(u) \in \partial\mathcal{A}_{x_0}(t_f)$, there exists a Lipschitz curve $p(t) \in T^*_{x(t,u)}$ for all $t \in [0,t_f]$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(x,p) = X_H(x,p,u),$$

$H(x(t),p(t),u(t)) = \max_{v\in U} H(x(t),p(t),v)$, *and* $p(t) \neq 0$, *for almost* $t \in [0,t_f]$.

The adjoint state $p(t_f)$ can be built through picking a normal co-vector to an approximation cone (Pontrjagin's approximation cone, see [4]) of the attainable set, and then pull-it back by the flow to obtain the whole curve. Many other proofs exist, mainly based on the so-called needle variations, one can be found in [18] for instance. The Pontrjagin's Maximum Principle (P.M.P.) can be seen as a consequence of this proposition - applied to the extended system - and for free final time, can be stated as follows (see [4] or [51] for the historical paper).

Theorem 1.2 (P.M.P.)
If (x, u) is an optimal pair for system (1.1.2), there exists a Lipschitz function $p(t) \in T^*_{x(t)}M$, $t \in [0, t_f]$, a constant $p^0 \leqslant 0$, $(p(t), p^0) \neq 0$, such that, for almost all t,

(i) (x, p) *is a solution of the Hamiltonian system associated with* $H(\cdot, \cdot, u(t))$*:*

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, u), \quad \dot{p} = -\frac{\partial H}{\partial x}(x, p, u),$$

(ii) $H(x(t), p(t), u(t)) = \max_{v \in U} H(x(t), p(t), v)$,

(iii) $H(x(t), p(t), u(t)) = 0$.

The adjoint state can be thought of as a Lagrange multiplier, and actually, up to multiplication with a scalar, we have $(p(t_f), p^0) = (P, P^0)$. Integral curves (x, p) of H satisfying (ii) and (iii) are called extremals. Their projections on M are extremal trajectories. As in classical Hamiltonian dynamics, and as consequence of the maximization condition, the pseudo-Hamiltonian evaluated along an extremal (x, p, u) is constant. Moreover, if the final time is free then this constant is zero.

Definition 1.3
The pair (p, p^0) is defined up to multiplication by a non-zero scalar. When p^0 is non-zero, we are in the normal *case, and one usually renormalize to $p^0 = -1$ or $-\frac{1}{2}$. If $p^0 = 0$, the extremal is called* abnormal.

Abnormal trajectories are one of the main difficulties in optimal control: it can be much harder to prove their optimality. The results of chapter I and II of this thesis apply to normal as well as abnormal extremals, as opposed to the ones in chapter III. There exist many versions of the the P.M.P., with fixed final time, for Bolza costs, and so on. For orbit transfer problems, it is convenient to work with boundaries which are not points, but submanifolds, namely, N_0,

$N_f \subset M$. The same result holds with transversality conditions on the adjoint state: $p(0) \perp T_{x_0} N_0$, $p(t_f) \perp T_{x_f} N_f$. The maximization condition allows us to calculate the optimal control, besides, if the maximum is in the interior of the set U, we have $\frac{\partial H}{\partial u}(x, p, u) = 0$. This works well for quadratic cost (energy minimization), however in this whole thesis, we will not be that lucky and the Hamiltonian will reach its maximum on ∂U (at least outside of the singularities).

Remark 1.1
Setting $H^{\max}(x, p) = \max_{u \in U} H(x, p, u)$, when well defined, allows one to work with a *true* autonomous Hamiltonian system, in the sense that it does not depend on the control. If $H^{\max}$ is regular enough (for instance, C^2 is enough), then extremals are just solutions of the differential equation given by $X_{H^{\max}}$.

As for time optimal problems, one just has to take $\varphi \equiv 1$, but then p^0 has no influence on the integral curves, and we can just work with the pseudo-Hamiltonian $H(x, p, u) = \langle p, f(x, u) \rangle$, with the condition $H(x(t), p(t), u(t)) \geqslant 0$, and $p(t) \neq 0$ for a.e. t. Normal extremal correspond to the $H > 0$ case, and abnormal ones to $H = 0$. In this chapter, we are mainly interested in the minimum time control of mechanical systems, namely, systems of the form

$$\ddot{q} + \nabla V(q) = u, \tag{1.1.4}$$

$V : O \to \mathbb{R}$, a smooth function defined on O an open subset of $\mathbb{R}^n$. A mechanical system is in particular a control affine system

$$\dot{x} = F_0(x) + \sum_{i=1}^{m} u_i F_i(x),$$

with $x = (q, v)$, $v = \dot{q}$, $F_0(x) = v . \frac{\partial}{\partial q} - \nabla V(q) . \frac{\partial}{\partial v}$ and the vector field supporting the control, in Cartesian coordinates are $F_i(x) = \frac{\partial}{\partial v_i}$. Their pseudo-Hamiltonian is affine on the control as well with $H(x, p, u) = H_0(x, p) + \sum_{i=1}^{m} u_i H_i(x, p)$ with $H_i(x, p) = \langle p, F_i(x) \rangle$. We will consider control bounded in an closed Euclidean ball $U = B(0, \varepsilon)$. In the next chapter, we will make a comparison with the case where the control lies in a polyhedron instead of a ball. Since $\frac{\partial H}{\partial u}(x, p, u)$ is everywhere non zero (except when all the H_i are vanishing), maximizing over all admissible value of u in the ball, we end up maximizing over its boundary: the sphere. We just proved

Proposition 1.2
The maximal Hamiltonian for minimum time affine control systems is

$$H^{\max}(x,p) = H_0(x,p) + \varepsilon\sqrt{\sum_{i=1}^{m} H_i(x,p)^2}.$$

Furthermore, the dynamical feedback on the control gives

$$u = \varepsilon \frac{(H_1,\dots,H_m)}{\|(H_1,\dots,H_m)\|}$$

which belongs to the m-sphere.

Clearly, our maximized Hamiltonian is only continuous, and irregularities of the dynamics occur when all the H_i are vanishing. In the next section, we will be interested in the behavior of the extremal flow and in the singularities of time minimization, thus, the parameter ε will not have any influence, and we will fix it to $\varepsilon = 1$. Let us see examples of some of the systems we will tackle.

Example 1.1
The controlled Kepler problem $\ddot{q} + \frac{q}{\|q\|^3} = u$ ($\|.\|$ denotes the Euclidean norm) in the plane models the motion of a spacecraft attracted by a fixed center. The control u is the thrust of its engine. Thus the maximized Hamiltonian is $H^{\max}(x,p) = p_q.v - p_v.\frac{q}{\|q\|^3} + \varepsilon\sqrt{p_{v_1}^2 + p_{v_2}^2}$.

Example 1.2
In the controlled planar restricted elliptic three body problem (RE3BP), the third body of negligible mass (a spacecraft or a satellite) is subjected to the attraction of two bodies (positions q^1 and q^2) in elliptic motion around their center of mass (Keplerian motion). The controlled dynamics is

$$\ddot{q} + \nabla V_\mu(t,q) = u, \tag{1.1.5}$$

with $V_\mu(t,q) = \frac{1-\mu}{|q-q^1(t)|} + \frac{\mu}{|q-q^2(t)|}$, the non-autonomous potential. Here, μ denotes the ratio of the masses of the two primaries. We end up with a non autonomous maximized Hamiltonian $H^{\max}(t,x,p) = p_q.v - p_v.\nabla V_\mu(t,q) + \varepsilon\sqrt{p_{v_1}^2 + p_{v_2}^2}$. The P.M.P. holds in that case, see [51].

Both of those examples are affine control problems with double input control (meaning, control vector of dimension 2).

1.2 Singularities of minimum time control for mechanical systems

We are interested in the optimal time control of affine control systems, more precisely, systems of the form

$$\dot{x} = F_0(x) + u_1 F_1(x) + u_2 F_2(x),$$

where the control $u = (u_1, u_2)$ is contained in some Euclidean ball and all vector fields are smooth, in a specific case detailed below, that contains mechanical systems. We also aim at developing a general theory and applying it to space mechanics, namely, to the controlled Kepler and restricted circular three-body problems. In this configuration, the controlled spacecraft (a satellite for instance) is under the influence of two primaries in circular motion. The mass of the satellite is supposed negligible with respect to the mass of the two primaries and the dynamics is

$$\ddot{q} + \nabla V_\mu(q) - 2J\dot{q} = u$$

where V_μ is the gravitational potential described in example 1.2, and fits our generic hypothesis (A) given in section 1.2.1. The control u here is the thrust of the engine. See [50] and [6] for more details on the restricted 3-body problem. In [20], the nilpotent case has been extensively treated and the averaged problem is considered. See also [15], where geodesic convexity is proved for the averaged system in the case $\mu = 0$ (Kepler). Here we will be interested in the original (as opposed to averaged) system. In this matter, the recent work [22] proved the non integrability of the extremal system in the Kepler case, and in [19], the L^1 minimization of the control is studied, and necessary and sufficient conditions for optimality are given. Recently, sufficient conditions for optimality have been also proved in [3] for a minimum time affine control system in a slightly different context. For the sake of simplicity, we carry out the arguments in a 4-dimensional manifold - which is the most relevant for our applications - but the method and result of section 1.2.2 can be adapted to a $2m$-dimensional manifold with an m-dimensional affine control.

In section 1.2.1, we start by recalling some classical results of geometric optimal control, with a particular emphasis on the Pontrjagin Maximum Principle, which reduces the problem to the study of a singular Hamiltonian system. The singularities of this system are related to the discontinuities of the optimal control u, also called switch. We study the local structure of the Hamiltonian flow under generic assumptions in section 1.2.2. The beginning of our study is built upon the analysis in [20] and goes one step further than the recent paper [2] where the flow is proved to be well-defined and continuous: Using the underlying normal hyperbolicity of the system, we provide a stratification such that the flow is smooth on each stratum. In section 1.2.3, we investigate the kind of singularity of the flow encountered when crossing strata. Thanks to a suitable normal form, we prove that the associated regular-singular transition results in a logarithmic term, implying it belongs to the log-exp category, [58]. We apply these results to the control of the circular restricted three-body problem in section 1.2.4. We finally investigate global properties of the flow and give upper bounds on the number of switchings of the control for this nonlinear system. Note that such bounds for time minimization are given in the linear case in [12]. In contrast with [23] where a subset of the switching set was studied, we treat here the general case using a comparison *à la Sturm*.

1.2.1 Setting

Let us first briefly introduce the general minimum-time affine control system

$$\dot{x} = F_0(x) + u_1 F_1(x) + u_2 F_2(x), \quad x \in M,\ u \in U \tag{1.2.1}$$

where U is the set of control values, i.e. unit Euclidean ball B of $\mathbb{R}^2$. We define the set of admissible controls: $\mathcal{U} = L^\infty([0,t_f],U)$. (One can equally let the admissible controls be in $L^1_{loc}([0,t_f],\mathbb{R}^2)$ rather than $L^\infty([0,t_f],\mathbb{R}^2)$, without much difference.)

The associated minimum-time control problem is:

$$\begin{cases} \dot{x}(t) = F_0(x(t)) + u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)), & t \in [0,t_f], \quad u \in U \\ x(0) = x_0 \\ x(t_f) = x_f \\ t_f \to \min . \end{cases} \tag{1.2.2}$$

By definition 1.2, the associated pseudo-Hamiltonian of (1.2.2) is

$$H_u(z) := H(x,p,u) = H_0(z) + u_1 H_1(z) + u_2 H_2(z),\ H_i(z) = \langle p, F_i(x)\rangle, \quad i = 0,1,2,$$

$z = (x,p) \in T_x^*M$.

We define the *singular locus*, or *switching surface*, as

$$\Sigma = \{z \in T^*M,\ H_1(z) = H_2(z) = 0\}.$$

Extremals along which (H_1, H_2) does not vanish are called *bang* arcs (u takes values in ∂B). An extremal is said to be *bang-bang* if it is a concatenation of bang arcs. Since we are interested in the behavior of the extremal flow, and its singularities, we can drop the parameter ε and consider the control in the unitary Euclidean ball. The following proposition is immediate from theorem 1.2 and proposition 1.2.

Proposition 1.3
An extremal lying out of Σ is an integral curve of the maximized Hamiltonian

$$H^{\max}(z) = H_0(z) + \sqrt{H_1^2(z) + H_2^2(z)},$$

and the associated control is

$$u = \frac{1}{\sqrt{H_1^2 + H_2^2}}(H_1, H_2)$$

(thus it lies in $\mathbb{S}^1$).

Extremals lying outside Σ are bang arcs and we will see that those crossing Σ are bang-bang extremals. We let $\bar{z} \in \Sigma$, and we are interested in the local behavior around $\bar{z}$. In what follows, we make the following assumption:

$$\det(F_1(\bar{x}), F_2(\bar{x}), F_{01}(\bar{x}), F_{02}(\bar{x})) \neq 0, \tag{A}$$

where $\bar{x} = \pi(\bar{z})$. This property is generic among vector fields and points of Σ, and holds, in particular, for control systems coming from mechanical systems. Namely, it means that brackets of order two with the drift generate the hole tangent space at each point. It is a sufficiency condition for global controllability, provided the drift F_0 is recurrent. Thus, for instance, controllability holds for the controlled Kepler problem and on some Hill's regions of the restricted circular three body problem (see [20]). Also, notice that since the adjoint vector cannot be zero, assumption (A) implies

$$H_1^2(\bar{z}) + H_2^2(\bar{z}) + H_{01}^2(\bar{z}) + H_{02}^2(\bar{z}) \neq 0$$

.

1.2.2 Stratification of the extremal flow

The result of this sections holds for $2n$-dimensional systems with n-dimensional control. We know after [20] that Σ can be partitioned in three subset, with three different local behavior for the flow:

$$\Sigma_- = \{H_{12}(z)^2 < H_{02}(z)^2 + H_{01}(z)^2\}$$
$$\Sigma_+ = \{H_{12}(z)^2 > H_{02}(z)^2 + H_{01}(z)^2\}$$
$$\Sigma_0 = \{H_{12}(z)^2 = H_{02}(z)^2 + H_{01}(z)^2\}.$$

In this paper, we treat the case Σ_-, which is the only one of importance in the applications to mechanical systems. The other cases (and more particularly, Σ_0) will be tackled in the next chapter. The main result of this section is the following.

Theorem 1.3
Assume (A) holds. Then there exists a neighborhood $O_{\bar{z}}$ of $\bar{z} \in \Sigma_-$ such that

(i) for every $\tilde{z} \in O_{\bar{z}}$ there exists a unique extremal $z(t, \tilde{z})$;

(ii) every extremal has at most one switch on $O_{\bar{z}}$;

(iii) extremals crossing Σ are bang-bang.

Furthermore, the local extremal flow $z : (t, z_0) \in [0, t_f] \times O_{\bar{z}} \mapsto z(t, z_0) \in T^*M$ is piecewise smooth when t_f is small enough. More precisely, it can be stratified as follows:

$$O_{\bar{z}} = S_0 \cup S^s \cup \Sigma$$

where S^s is the codimension-one submanifold of initial conditions leading to the switching surface, $S_0 = O_{\bar{z}} \backslash (S^s \cup \Sigma)$. Both are stable by the flow, which is smooth on $[0, t_f] \times S_0$, and on $[0, t_f] \times S^s \backslash \Delta$ where $\Delta = \{(\bar{t}(z_0), z_0),\ z_0 \in S^s\}$, and $\bar{t}(z_0)$ is the switching time of the extremal initializing at z_0, and continuous on $O_{\bar{z}}$.

In a slow enough time, so that we have sufficient regularity to define stable and unstable manifolds to Σ, S^s is actually the global stable manifold. Before going into the proof, let us see a simple example given by a nilpotent approximation (only brackets of length smaller than 2 are non zero) of the two body controlled problem (see section 1.2.4). See [32] for a detailed presentation of nilpotent approximations of an affine control system.

Example 1.3
The nilpotent approximation [20], is

$$\begin{cases} \dot{x}_1 = 1 + x_3 & \dot{x}_3 = u_1 \\ \dot{x}_2 = x_4 & \dot{x}_4 = u_2 \end{cases} \tag{1.2.3}$$

with the same bound on the control $|u| \leqslant 1$. The the maximized Hamiltonian is:

$$H(x,p) = p_1(1+x_3) + p_2x_4 + \sqrt{p_3^2+p_4^2}$$

(p_i being the adjoint variable of x_i), and the codimension two submanifold

$$\Sigma = \{p_3 = 0\} \cap \{p_4 = 0\} = \Sigma_-$$

is the singular locus, because F_{12} is null here. Notice that p_1 and p_2 are constant. Let $a = -p_1(0)$ and $c = -p_2(0)$. We then get $p_3(t) = at+b$, $p_4(t) = ct+d$, with $b = p_3(0)$ et $d = p_4(0)$. From (1.2.3), the regularity of the flow is given by the x_3 and x_4 components. We have

$$\dot{x}_3 = \frac{at+b}{\sqrt{(at+b)^2+(ct+d)^2}}, \tag{1.2.4}$$

thus we see that singularities arise when $t \mapsto (at+b, ct+d)$ vanishes, ie, $ad-bc = 0$ which defines our codimension one submanifold $S^s = \{p_1p_4 - p_2p_3 = 0\} \backslash \{p = 0\}$ (remember that the adjoint state cannot be zero for a minimum-time extremal thanks to the Maximum Principle). Naturally, we get a symmetric dynamics for x_4 and end up with the same sub-manifold. One can notice that this strata is stable by the flow of H: If $z_0 \in S^s$, $z(t,z_0) = (x(t),p(t)) \in S^s$. Outside S^s, we can explicitly solve (1.2.4), to obtain:

$$\begin{aligned} x_3(t,z_0) &= \frac{a}{a^2+c^2}\left(\sqrt{(at+b)^2+(ct+d)^2} - \sqrt{b^2+d^2}\right) \\ -c\frac{ad-bc}{(a^2+c^2)^{3/2}}&\left[argsh\left(\frac{(a^2+c^2)t+ab+cd}{ad-bc}\right) - argsh\left(\frac{ab+cd}{ad-bc}\right)\right] + x_3(0). \end{aligned} \tag{1.2.5}$$

It becomes then obvious that the flow of the nilpotent approximation is smooth outside S^s. If a and c are null, p_3 and p_4 become constant, and since p cannot vanish, switching never occurs. Now observe that the flow can be continuously extended to S^s by

$$x_3(t,z_0) = \frac{a}{a^2+c^2}\left(\sqrt{(at+b)^2+(ct+d)^2} - \sqrt{b^2+d^2}\right) + x_3(0)$$

for all $z_0 \in S^s \backslash \{p = 0\}$. Restricted to S^s, the flow is locally smooth, except at switches. We also have global continuity, even-though the flow is not Lipschitz continuous. Furthermore, on this simple model a singularity of the type "$z \ln z$" appears when crossing S^s. This assumption is the subject of section 1.2.3.

Proof. Let us now prove theorem 2.1. Consider a local chart in $O_{\bar{z}}$,

$$z \in T^*M \mapsto (x, p_1, p_2, p_3, p_4) \in \mathbb{R}^4 \times \mathbb{R}^4 \mapsto (x, H_1, H_2, H_{01}, H_{02}).$$

This map is a smooth change of coordinates according to assumption (A). Then, a polar blow-up is used to study the dynamics near the singularity by setting

$$(H_1, H_2) = (\rho \cos\theta, \rho \sin\theta), \ (\rho, \theta) \in \mathbb{R} \times \mathbb{S}^1.$$

In polar coordinates we have an expression for the control $u = (\cos\theta, \sin\theta)$, and $\Sigma = \{\rho = 0\}$. At this point, the dynamics is the following:

$$\begin{cases} \dot{x} = F_0(x) + \cos\theta F_1(x) + \sin\theta F_2(x) \\ \dot{\rho} = \cos\theta H_{01} + \sin\theta H_{02} \\ \dot{\theta} = \frac{1}{\rho}(H_{12} + \cos\theta H_{02} - \sin\theta H_{01}) \\ \dot{H}_{01} = H_{001} + \cos\theta H_{101} + \sin\theta H_{201} \\ \dot{H}_{02} = H_{002} + \cos\theta H_{102} + \sin\theta H_{202} \end{cases} \tag{1.2.6}$$

The vector fields F_i are smooth, and then, so are the H_i and all their Poisson brackets. Setting a new time to desingularize $dt = \rho ds$ we get:

$$\begin{cases} x' = \rho(F_0(x) + \cos(\theta)F_1(x) + \sin(\theta)F_2(x)) \\ \rho' = \rho(\cos\theta H_{01} + \sin\theta H_{02}) \\ \theta' = H_{12} + \cos(\theta)H_{02} - \sin(\theta)H_{01} \\ H'_{01} = \rho(H_{001} + \cos\theta H_{101} + \sin\theta H_{201}) \\ H'_{02} = \rho(H_{002} + \cos\theta H_{102} + \sin\theta H_{202}), \end{cases} \tag{1.2.7}$$

while the notation $'$ denotes the derivation with respect to time s. In this new time, the autonomous vector field is smooth, which implies existence and uniqueness of maximal solutions through a point , as well as smoothness of the flow. We will denote by X the vector field of (1.2.7). Note that when ρ is null, only θ is non constant, in particular $\Sigma = \{\rho = 0\}$ is invariant by the flow. Besides, we have the following formula:

$$\operatorname{ad}\rho = \operatorname{ad} H_0 + \cos\theta \operatorname{ad} H_1 + \sin\theta \operatorname{ad} H_2. \tag{1.2.8}$$

In the following we will denote $\pi(\bar{z}) = \bar{x}$, $H_{ij}(\bar{z}) = \bar{H}_{ij}$, $i, j = 0, 1, 2$. The next lemma establishes the the crucial following fact: In each part of Σ, the derivative with respect to θ has a different number of equilibria:

Lemma 1.1

(i) $\forall z \in \Sigma_-$, $\theta \mapsto H_{12}(z) + \cos(\theta)H_{02}(z) - \sin(\theta)H_{01}(z)$ *has two zeros, denoted by* θ_- *and* θ_+*. The map* $y = (x, H_{01}, H_{02}) \in Y \cap \Sigma \mapsto \theta_\pm(y)$ *is smooth and well defined, where* Y *is a small neighborhood of* $(\bar{x}, \bar{H}_{01}, \bar{H}_{02})$*.*

(ii) $\forall z \in \Sigma_+$, $\theta \mapsto H_{12}(z) + \cos(\theta)H_{02}(z) - \sin(\theta)H_{01}(z)$ *has no zero.*

(iii) $\forall z \in \Sigma_0$, $\theta \mapsto H_{12}(z) + \cos(\theta)H_{02}(z) - \sin(\theta)H_{01}(z)$ *has exactly one zero.*

Proof of Lemma 1.1. Indeed, with polar coordinates on the brackets

$$(H_{01}, H_{02}) = r(\cos\psi, \sin\psi)$$

we get $H_{12}(z) + \cos(\theta)H_{02}(z) - \sin(\theta)H_{01}(z) = H_{12}(z) - r\sin(\theta - \psi)$ and conclude by noting that $H_{12}(z)/r = \sin(\theta - \psi)$ has two solutions, θ_- and θ_+, if $\bar{z} \in \Sigma_-$, no solution if $\bar{z} \in \Sigma_+$ and one if $\bar{z} \in \Sigma_0$ (since $H_{12}(z)/r = \pm 1$). To check that we can use the implicit function theorem, denote

$$g : (x, \theta, H_{01}, H_{02}) \in \Sigma \mapsto H_{12}(z) + \cos(\theta)H_{02}(z) - \sin(\theta)H_{01}(z)$$

and notice that

$$\frac{\partial g}{\partial \theta}(y, \theta_\pm) = -r\cos(\theta_\pm - \psi) = \pm\sqrt{r(z)^2 - H_{12}(z)^2} \neq 0$$

for $y \in Y$. $\square$

Let us briefly recall some notions about normal hyperbolicity. Let us endow M with a Riemannian metric, with associated norm $\|.\|$.

Definition 1.4

A diffeomorphism $f : M \circlearrowleft$ *is said to be normally hyperbolic along a compact submanifold* N *if* N *is invariant by* f *and the tangent bundle of* M *along* N *has a splitting* $T_zM = E(z)^u \oplus T_zN \oplus E(z)^s$, $z \in N$, *such that* $df\, E^{u,s}(x) = E^{u,s}(f(x))$ *(*f *preserves the splitting), and there exists* $\lambda_1 \leqslant \mu_1 < \lambda_2 \leqslant \mu_2 < \lambda_3 \leqslant \mu_3$, *with* $\mu_1 < 1 < \lambda_3$, *such that*

$$\lambda_1 \leqslant \|df_{|E^s}\| \leqslant \mu_1,\ \ \lambda_2 \leqslant \|df_{|TN}\| \leqslant \mu_2,\ \ \lambda_3 \leqslant \|df^{-1}_{|E^u}\| \leqslant \mu_3. \tag{1.2.9}$$

This property can be described by saying that the contraction (resp. expansion) in the the stable (resp. unstable) direction is stronger than tangentially to N. The distributions E^s and E^u turn out to be locally integrable and one can construct the local stable and unstable manifolds, $W(z)^s$ and $W(z)^u$ respectively tangent to $E^s(z)$ and $E^u(z)$ at each point $z \in N$. Also, define $W^{so} = \bigcup_{z \in N} W^s(z)$ and $W^{uo} = \bigcup_{z \in N} W^u(z)$ the local stable (resp. unstable) manifold of N. Define also l_s, l_u as the biggest integers such that $\mu_1 \leqslant \lambda_2^{l_u}$ and $\mu_2^{l_s} \leqslant \lambda_3$.

We now recall a theorem of Hirsch, Pugh and Shub (theorem 3.5 in [33], see also [47]) giving the regularity of the manifold in terms of the ratio of the contraction and expansion rates.

Theorem 1.4 (Hirsch, Pugh, Shub)
Any f-invariant submanifold which is close enough to N is included in $W^{so} \cup W^{uo}$. Furthermore, W^{so} and W^{uo} are smooth submanifolds of class C^{l_s} and C^{l_u} respectively.

In our case, we have two codimension-two submanifolds of equilibrium points, namely $\bar{z}_+ = (x, 0, \theta_+(y), H_{01}, H_{02})$ and $\bar{z}_- = (x, 0, \theta_-(y), H_{01}, H_{02})$. We set

$$\cos_- \theta H_{01} + \sin \theta_- H_{02} = -\sqrt{r^2 - H_{12}^2} < 0$$

(so the value for θ_+ will be the opposite). The Jacobian of the system (1.2.7) has two non-zero eigenvalues at those points:

$$\cos \theta_\pm H_{01} + \sin \theta_\pm H_{02}$$

and their opposite, and a 6-dimensional kernel. Given the spectrum of the Jacobian in $\bar{z}_\pm$ we have a unidimensional stable submanifold $W^s(\bar{z}_\pm)$, and a unidimensional unstable submanifold $W^u(\bar{z}_\pm)$ in each equilibria $\bar{z}_\pm$. The flow is thus normally hyperbolic to the manifold $N = \{z_-(x, 0, \theta_-(y), H_{01}, H_{02}),\ z_- \in O_{\bar{z}}\}$: The tangent space is split as in Definition 1.4, with λ_2 and μ_2 being 1. On N the dynamics is trivial: every point is an equilibrium. Hence there exists a unique trajectory converging to $\bar{z}_-$ (in infinite time s) in the stable manifold $W^s(\bar{z}_-)$. On Σ however, everything is constant except θ, which makes a heteroclinic connexion from θ_- to θ_+. Then, an extremal will converge to $\bar{z}_+$ when $s \to -\infty$. Since $\rho' = \rho(\cos \theta H_{01} + \sin \theta H_{02}) < 0$, in $\bar{z}_-$ there could be no extremal going out of Σ, since $\rho \geqslant 0$ is preserved. The situation is symmetric in $\bar{z}_+$, no extremal can converge to Σ in $\bar{z}_+$. The system (1.2.7) is autonomous, so there is a unique extremal passing through $\bar{z}$, besides, this is happening in finite time t. Indeed, notice that $z \mapsto \cos \theta H_{01}(z) + \sin \theta H_{02}(z) = -\sqrt{r^2(z) - H_{12}^2(z)}$ is smooth, and as such, bounded on $O_{\bar{z}}$. It is also negative, and note $C < 0$ a negative upper bound. Now by

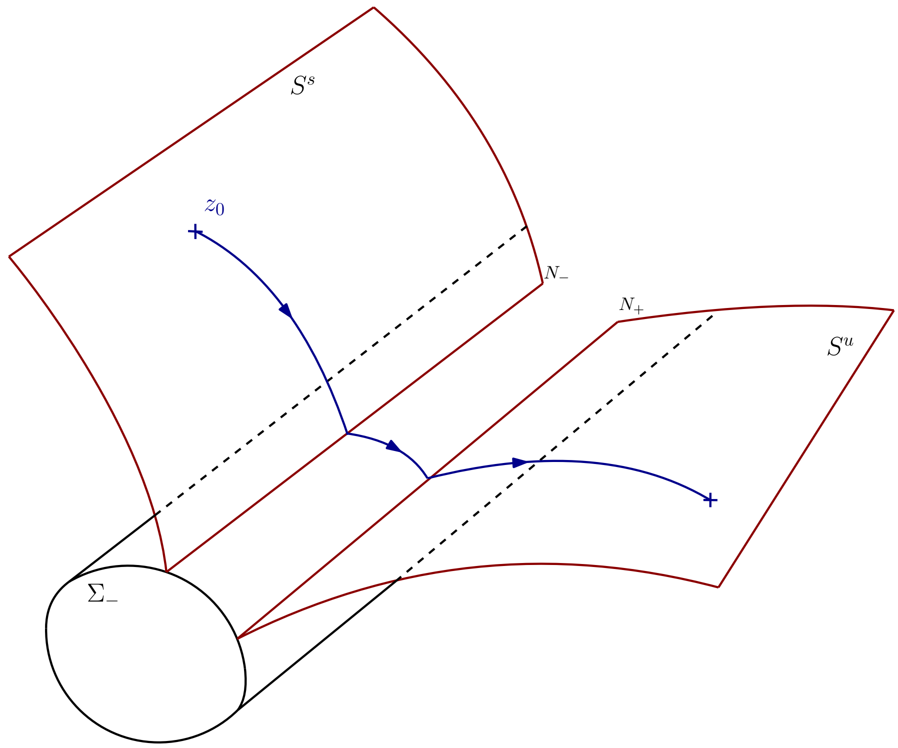

Figure 1.1: Stratification of the flow into regular submanifolds.

considering the dynamics of ρ, see that $\dot{\rho}(s) \leqslant \rho(s)C$. Hence, by Gronwall's lemma $\rho(s) < \rho(0)e^{Cs}$. Now the time interval before reaching Σ satisfies

$$\Delta t < \int_0^\infty \rho(s)ds < \rho(0)/C < \infty.$$

In what follows we investigate the regularity of the flow.

Lemma 1.2
There exists a codimension one submanifold $S^s = \bigcup_{z \in N} W^s(z)$—the set of initial conditions leading to Σ—preserved by the flow, and on which it is locally smooth. More precisely, the map $z : [0, t_f] \times S^s \backslash \Delta \to O_{\bar{z}}$ is smooth, where $\Delta = \{(\bar{t}(z_0), z_0),\ z_0 \in S^s\}$, $\bar{t}(z_0)$ being the switching time of the extremal passing through z_0.

Proof of Lemma 1.2. We begin the proof by showing that S^s is a well defined codimension 1 submanifold. Recall that the Jacobian of the vector field has a convenient spectrum on N: A 6-dimensional kernel, and two eigenspaces with opposite eigenvalues

$$\cos\theta_- H_{01} + \sin\theta_- H_{02} = -\sqrt{r^2 - H_{12}^2},$$

and $+\sqrt{r^2 - H_{12}^2}$. As such, the flow is normally hyperbolic to N, then $V = \bigcup_{z\in N} W^s(z)$ is the so-called stable manifold of N. V is smooth by theorem 1.4 since there is no dilatation nor contraction on TN. Then the submanifold we are looking for is $S^s = V \cap \{\rho > 0\}$, and $\dim S^s = \dim N + \dim W^s(z) = 7$. It remains to show that the map $z : (t, z_0) \in [0, t_f] \times S^s \backslash \Delta \mapsto z(t, z_0) \in O_{\bar{z}}$ is smooth. Let's set $\bar{t}(z_0) = \int_0^\infty \rho(s, z_0)ds$ the contact time with the singularity Σ, we know that $\bar{t}(z_0) < \infty$. The flow is smooth in the time s, hence $z_0 \mapsto \rho(s, z_0)$ is smooth and by the classical dominated convergence argument, $\bar{t}(z_0)$ is smoothly depending on $z_0 \in O_{\bar{z}}$. Obviously $z : (t, z_0) \in [0, \bar{t}(z_0)) \times S^s \mapsto z(t, z_0) \in O_{\bar{z}}$ is smooth. By uniqueness $z(t, z_0) = z(t - \bar{t}(z_0), z(\bar{t}(z_0), z_0))$, and we use the fact that the contact point with the singular locus Σ is depending smoothly on the initial condition on S^s, more precisely, the map $z_0 \in S^s \mapsto z(\bar{t}(z_0), z_0)$ is smooth. It is straightforward by writing $z(\bar{t}(z_0), z_0) = \int_0^\infty X(z(s, z_0))ds + z_0$. The integrand is bounded uniformly with respect to z_0, indeed, $s \mapsto z(s, z_0)$ is only parameterizing the stable manifold $W^s(z(\bar{t}(z_0), z_0)) \subset O_{\bar{z}}$, but this neighborhood is relatively compact (and independent of z_0), and the vector field X is bounded on $O_{\bar{z}}$. Thus this integral is smooth with respect to the parameter z_0. Now notice that $(t, z_0) \mapsto z(t - \bar{t}(z_0), z(\bar{t}(z_0), z_0)) = z(t, z_0)$ is an extremal initializing on Σ at time $\bar{t}(z_0)$. By the symmetry $t \mapsto -t$, this situation is analogous to the previous one, this map is smooth as long as $t \neq \bar{t}(z_0)$, which conclude the proof of Lemma 1.2. □

We now know that the flow is smooth outside of S^s and restricted to S^s, we also know that it is Lipschitz with respect to the time t. It remains to prove its global continuity on $O_{\bar{z}}$. Let z_0, $z_1 \in O_{\bar{z}}$, t, t' in $[0, t_f]$ and $O_\delta \subset O_{\bar{z}}$ be a small neighborhood of $\bar{z}$. Thanks to the previous assumption, we can assume without loss of generality, that $z_0 \in S^s$, $z_1 \in O_{\bar{z}} \backslash S^s$ and that the extremal from z_0 is passing throw $\bar{z}$. We would like to control de quantity $|z(t, z_0) - z(t', z_1)|$. Let $\varepsilon > 0$, and note t_0, the contact time with O_δ, and t_0' the exit time from this neighborhood: Namely $t_0 = \inf\{t \in \mathbb{R},\ \text{s. t. } z(t, z_0) \in O_\delta\}$. If z_0 and z_1 are close enough, $|z(t_0, z_0) - z(t_0, z_1)| < \varepsilon/2$; simply because the flow is continuous when the singular locus is not crossed yet. We will use the following Lemma to conclude:

Lemma 1.3 ([2])
For all $\delta > 0$ there exists a neighborhood O_δ of $\bar{z}$ in which every extremal spends a time interval uniformly (that is not depending on the extremal) smaller than δ.

Proof of Lemma 1.3. We will prove it in a neighborhood $O_{\bar{z}_-}$ of $\bar{z}_-$, the situation being symmetric around $\bar{z}_+$, and an extremal in Σ spending 0 time t. Let's define $O_\delta = \{z \in O_{\bar{z}_-}, \rho < \delta, |\theta - \theta_-| < \delta\}$, $z \mapsto \cos(\theta)H_{01}(z) + \sin(\theta)H_{02}(z)$ is smooth and thus bounded on O_δ. Now set

$$M_\delta = \sup_{z \in O_\delta} \cos(\theta)H_{01}(z) + \sin(\theta)H_{02}(z),$$

remember it is negative on $O_{\bar{z}_-}$. Then, for any extremal in O_δ, $\frac{\dot{\rho}(s)}{\rho(s)} \leqslant M_\delta$, which implies

$$\rho(s) < \rho(0)e^{M_\delta s}.$$

So that, we can control its time interval in 0_δ by

$$\Delta_{O_\delta} t < \int_0^\infty \rho(0)e^{M_\delta s} ds = -\frac{R}{M_\delta},$$

this quantity tends to 0 when R does, and the lemma is proved. Then, with a good choice of O_δ, $|z(t_0', z_0) - z(t_0, z_0)| \leqslant \varepsilon/2$, and $|z(t_0', z_0) - z(t_0, z_1)| \leqslant |z(t_0', z_0) - z(t_0, z_0)| + |z(t_0, z_0) - z(t_0, z_1)| \leqslant \varepsilon$. Now notice that, $z(t, z_0) = z(t - t_0', z(t_0', z_0))$, and we use the regularity of the system when the singular locus is not crossed to conclude. $\square$

Remark 1.2
In the case $\bar{z} \in \Sigma_-$, we can quantify the jump on the control at a switching time $\bar{t}$ in terms of Poisson brackets:

$$u(\bar{t}_\pm) = (\cos\theta_\pm, \sin\theta_\pm) = \frac{1}{r^2}(-H_{02}H_{12} \pm H_{01}\sqrt{r^2 - H_{12}^2}, H_{01}H_{12} \pm H_{02}\sqrt{r^2 - H_{12}^2}).$$

Regarding the corresponding extremal trajectory, the jump is then given by

$$\dot{x}(\bar{t}_+) - \dot{x}(\bar{t}_-) = \frac{2\sqrt{r^2 - H_{12}^2}}{r^2}(H_{01}(\bar{z})F_1(\bar{x}) + H_{02}(\bar{z})F_2(\bar{x})).$$

As said above, theorem 2.1 still holds in higher dimension. The same proof can be adapted with a little bit more technicalities by taking spherical coordinates instead of polar ones when doing the blow up.

1.2.3 Regular-singular transition

In our applications, and more generally whenever the distribution is involutive, the interesting case is Σ_-, and we have the stratification defined in the previous section. This stratification of the flow raises the question of the transition: How does the flow behave when one is getting close to the stratum S^s? We answer that question by considering the Poincaré map between two well chosen sections. Recall the dynamics in the time s:

$$\begin{cases} x' = \rho(F_0(x) + cos(\theta)F_1(x) + \sin(\theta)F_2(x)) \\ \rho' = \rho(\cos\theta H_{01} + \sin\theta H_{02}) \\ \theta' = H_{12} + \cos(\theta)H_{02} - \sin(\theta)H_{01} \\ H'_{01} = \rho(H_{001} + \cos\theta H_{101} + \sin\theta H_{201}) \\ H'_{02} = \rho(H_{002} + \cos\theta H_{102} + \sin\theta H_{202}). \end{cases}$$

in polar coordinates on the Poisson brackets: $(H_{01}, H_{02}) = r(\cos\phi, \sin\phi)$ one gets:

$$\begin{cases} \rho' = r\rho\cos(\theta - \phi) \\ \theta' = H_{12} - r\sin(\theta - \phi) \\ \xi' = \rho h(\rho, \theta, \xi) \end{cases} \tag{1.2.10}$$

where $\xi = (x, r, \phi)$ and h is a smooth function. We can set $\psi = \theta - \phi$, rescale the time according to $\mathrm{d}v = r\mathrm{d}s$, and study a system with the following structure (the derivation still being noted $'$):

$$\begin{cases} \rho' = \rho\cos\psi \\ \psi' = g(\rho, \psi, \xi) - \sin\psi = G(\rho, \psi, \xi) \\ \xi' = \rho h(\rho, \psi, \xi) \end{cases} \tag{1.2.11}$$

where g, h are smooth functions defined on an open set O of $\mathbb{R} \times \mathbb{R} \times D$, D being a compact domain of $\mathbb{R}^k$; h has values in $\mathbb{R}^k$ and $g(\rho, \psi, \xi) = a(\xi) + \rho b(\xi, \psi) + O(\rho^2)$ and $|g| < 1$ on O. (This comes from the fact that H_{12} is a smooth function in $(\rho\cos\theta, \rho\sin\theta)$.) Equilibria occur when $\rho = 0$, $G = 0$ and are semi-hyperbolic, since they are outside $\{\psi = \pm\pi/2\}$ (case Σ_-). More precisely, it was shown in the last section that the flow of this system is normally hyperbolic to the manifold $\{\rho = 0\} \cap \{G = 0\}$. For each ξ, there are two equilibria $z_\pm$. Thanks to the structure of g, we get $\frac{\partial g}{\partial\psi}(0, \psi, \xi) = 0$, thus $\frac{\partial G}{\partial\psi}(0, \psi, \xi) = -\cos\psi \neq 0$. $\{G = 0\} \cap \{\rho = 0\}$ can then be parameterized by the map $\xi \mapsto \psi(\xi)$ by the implicit function theorem. Equilibria are then given by $z_{\pm} = (0, \psi_{\pm}(\xi), \xi)$ coordinates, and define two codimension two submanifolds. In each equilibria, the stable and unstable manifolds are of dimension one. Their union form a codimension one submanifold, $S^s = \bigcup_\xi W^s(z_-)$, (resp. $S^u = \bigcup_\xi W^u(z_+)$) is thus the submanifold of initial condition leading to (resp. in negative time) an equilibrium. The aim of the following work is to study the regular-singular transition, or more precisely, the type of singularity occurring when one crosses S^s.

Introducing $\omega = \psi - \psi(\xi)$ along $\{G = 0\}$ (the analysis is similar for ψ_2 on the unstable manifold), we will study (1.2.11) near $\omega \sim 0$. The change of coordinates $(\rho, \psi, \xi) \mapsto (\rho, \omega, \xi)$ gives

$$\begin{cases} \rho' = \rho\cos(\omega + \psi(\xi)) \\ \omega' = g(\rho, \omega + \psi(\xi), \xi) - \sin(\omega + \psi(\xi)) - \rho\frac{\partial\psi}{\partial\xi}(\xi).h(\rho, \omega + \psi(\xi), \xi) \\ \xi' = \rho h(\rho, \omega + \psi(\xi), \xi). \end{cases}$$

As g has the form given by (ii), $g(\rho, \omega + \psi(\xi), \xi) = a(\xi) + \rho b(\omega + \psi(\xi), \xi) + O(\rho^2)$, and $g(0, \psi(\xi), \xi) = \sin(\psi(\xi)) = a(\xi)$. So that system (1) is equivalent to

$$\begin{cases} \rho' = \lambda(\xi)\rho(1 + O(\omega^2)) \\ \omega' = \beta(\xi)\rho - \lambda(\xi)\omega + O((\rho + |\omega|)^2) \\ \xi' = \rho(\gamma(\xi) + O(\rho + |\omega|)) \end{cases} \tag{1.2.12}$$

with $\lambda(\xi) = \cos(\psi(\xi))$ and β, γ smooth functions (depending on the derivatives of g, h). The Jacobian matrix of (1.2.12) is

$$\begin{pmatrix} \lambda(\xi) & 0 & 0 \\ \beta(\xi) & -\lambda(\xi) & 0 \\ \gamma(\xi) & 0 & 0 \end{pmatrix}$$

Let us now find a change of coordinates making this Jacobian diagonal: $(\rho, \omega, \xi) \mapsto (\rho, \tilde\omega, \tilde\xi) = (\rho, \omega + A(\xi)\rho), \xi + B(\xi)\rho)$. We get $\tilde\omega' = \omega' + \frac{\partial A}{\partial\xi}(\xi).\xi'\rho + A(\xi)\rho' = \omega' + A(\xi)\rho + O(\rho^2)$. Thus $\tilde\omega' = (\beta(\xi) + 2A(\xi)\lambda(\xi))\rho - \lambda(\xi)\tilde\omega + O((\rho + |\omega|)^2)$ and by picking $A = -\frac{\beta}{\lambda}$ we obtain what we were looking for: Indeed, with this change of variables, $O((\rho + |\omega|)^k) = O((\tilde\rho + |\tilde\omega|)^k)$ for all k. Now, $\tilde\xi' = \xi' + \frac{\partial A}{\partial\xi}(\xi).\xi'\rho + B(\xi)\rho' = \rho(\gamma(\xi) + B(\xi)\lambda(\xi)) + O((\rho + |\omega|)^2)$ and we pick $B(\xi) = -\frac{\gamma}{\lambda}$. Slightly abusing of the notations (except for ρ), we still note the new variables

(ρ, ω, ξ) and obtain the new vector field

$$\begin{cases} \rho' = \lambda(\xi)\rho(1 + O(\rho)) \\ \omega' = -\lambda(\xi)\omega + O((\rho + |\omega|)^2) \\ \xi' = \rho O(\rho + |\omega|). \end{cases} \tag{1.2.13}$$

System (1.2.13) is smoothly equivalent to

$$Y : \begin{cases} \rho' = -\rho(1 + O(\rho)) \\ \omega' = \omega + O((\rho + |\omega|)^2) \\ \xi' = \rho O(\rho + |\omega|). \end{cases} \tag{1.2.14}$$

The Jacobian of this system is diagonal and even constant. Let us now state a smooth normal form theorem:

Proposition 1.4 ($\mathcal{C}^\infty$-normal form)
Set $u = \rho\omega$, then there exist smooth functions A, B, C on a neighborhood of $D \times 0_u$ such that Y is equivalent to

$$Y^\infty : \begin{cases} \rho' = -\rho(1 + uA(u, \xi)) \\ \omega' = \omega(1 + uB(u, \xi)) \\ \xi' = uC(u, \xi) \end{cases} \tag{1.2.15}$$

Under the corresponding change of coordinates, the globally invariant manifold S^s, fibered by stable manifolds, becomes $\{\omega = 0\}$, and the equilibria are $\{(0, 0, \xi)\}$. We can now make a precise statement: For given ρ_0 et ω_f, both positive, consider the two sections $\Pi_0 \subset \{\rho = \rho_0\}$ and $\Pi_f \subset \{\omega = \omega_f\}$. As Π_0 is transverse to $\{\omega = 0\}$, it can be parameterized by (ω, ξ) coordinates. Similarly, Π_f is transverse to $\{\rho = 0\}$ and can be parameterized by (ρ, ξ) coordinates.

Theorem 1.5
Let $T : \Pi_0 \to \Pi_f$ be the Poincaré mapping between the two sections, $T(\omega_0, \xi_0) = (\rho(\omega_0, \xi_0), \xi(\omega_0, \xi_0))$. Then, T is a smooth function in $(\omega_0 \ln \omega_0, \omega_0, \xi_0)$ as there exist smooth functions R and X defined on a neighborhood of $\{0\} \times \{0\} \times D$ such that

$$T(\omega_0, \xi_0) = (R(\omega_0 \ln \omega_0, \omega_0, \xi_0), X(\omega_0 \ln \omega_0, \omega_0, \xi_0)).$$

Thus, T belongs to the log-exp category.

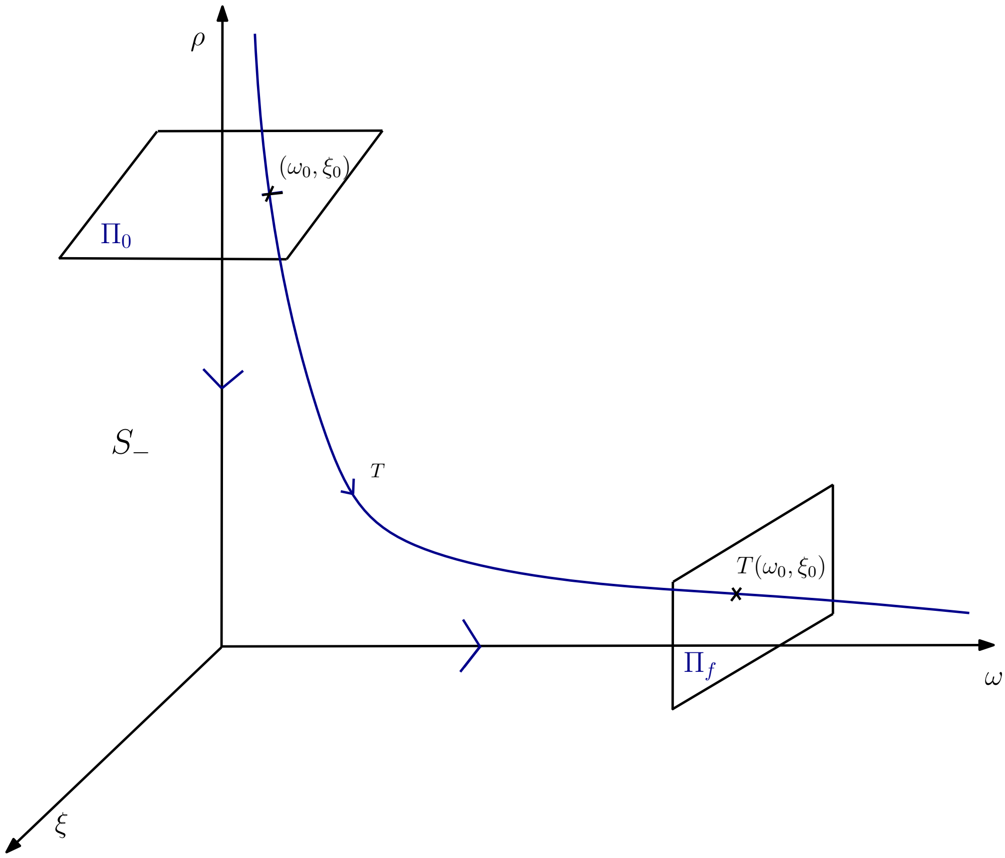

Figure 1.2: Transition map between the two sections

One can look once again at the example of section 2.2, the nilpotent approximation for the minimum time Kepler problem. Recall we had

$$\begin{aligned} x_3(t, z_0) &= \frac{a}{a^2+c^2}\left(\sqrt{(at+b)^2+(ct+d)^2}-\sqrt{b^2+d^2}\right) \\ -c\frac{ad-bc}{(a^2+c^2)^{3/2}}&\left[\operatorname{argsh}\left(\frac{(a^2+c^2)t+ab+cd}{ad-bc}\right)-\operatorname{argsh}\left(\frac{ab+cd}{ad-bc}\right)\right]+x_3(0) \end{aligned} \tag{1.2.16}$$

giving the regularity. When one crosses Σ the determinant $ad-bc$ becomes 0, and one indeed gets singularities of the form "$z \ln z$". Before proving the normal form result, let us demonstrate how it implies Theorem 1.5.

Proof of Theorem 1.5. First, note that system (1.2.15) is equivalent to

$$\begin{cases} \omega' = \omega \\ \rho' = -\rho(1 + uA(u,\xi)) \\ \xi' = uC(u,\xi) \end{cases} \tag{1.2.17}$$

with A standing now for $\frac{A-B}{1+uB}$, and C for $\frac{C}{1+uB}$. It has the same trajectories, and thus the same Poincaré mapping between the two sections. The transition time is given by the first equation: $s(\omega_0) = \ln(\omega_f/\omega_0)$. (The singular-regular transition occurs when $\omega_0 \to 0$, and the transition time tends to infinity.) Still denoting $u = \rho\omega$, (1.2.17) implies

$$\begin{cases} u' = -u^2 A(u,\xi), \\ \xi' = uC(u,\xi), \end{cases} \tag{1.2.18}$$

that we want to integrate from an initial condition on Π_0 in time $s(\omega_0)$. We extend this system by the trivial equation $\omega_0' = 0$, and denote φ its associated flow. Then, $T(\omega_0,\xi_0) = \varphi(\ln(\omega_f/\omega_0), \omega_0, \rho_0\omega_0, \xi_0)$ (on Π_0, $u_0 = \rho_0\omega_0$). It is not the form we are looking for since $\ln(\omega/\omega_0)$ is not regular at $\omega_0 = 0$.
Let us now make a change of time and consider the following rescaled system:

$$\tilde{Z} \begin{cases} \omega_0' = 0, \\ u' = -(u^2/\omega_0)A(u,\xi), \\ \xi' = (u/\omega_0)C(u,\xi). \end{cases} \tag{1.2.19}$$

For $\omega_0 > 0$, its flow $\tilde{\varphi}$ is well defined and the Poincaré mapping is obtained by evaluating it in time $\omega_0 \ln(\omega_f/\omega_0)$,

$$T(\omega_0,\xi_0) = \tilde{\varphi}(\omega_0 \ln(\omega_f/\omega_0), \omega_0, \rho_0\omega_0, \xi_0).$$

However, $\tilde{\varphi}$ is not even defined in $\omega_0 = 0$: we use the blow up on $\{u = \omega = 0\}$ to prove that T has the required form. Set $f(u,\omega,\xi) = (\eta,\omega,\xi)$ with $\eta = u/\omega$: In coordinates (ω,η,ξ), the pulled back system writes

$$\hat{Z} : \begin{cases} \omega_0' = 0, \\ \eta' = -\eta^2 A(\eta\omega_0,\xi), \\ \xi' = \eta\, C(\eta\omega_0,\xi). \end{cases} \tag{1.2.20}$$

The vector field $\hat{Z} = f_*^{-1}(\tilde{Z})$ is actually smooth and denote $\hat{\varphi}$ its - also smooth - flow. The blow up map f send a cone $-\eta_0\omega_0 \leqslant u \leqslant \eta_0\omega_0$, $\omega_0 \in [-\omega_1,\omega_1]$, on a rectangle $-\eta_0 \leqslant \eta \leqslant \eta_0$, $-\omega_1 \leqslant \omega_0 \leqslant \omega_1$. As $\hat{\varphi} = (\hat{\eta}, \hat{\xi})$ is smooth on the band $\omega_0 \in [-\omega_1, \omega_1]$, $\eta_0 \in [-M, M]$, $\xi \in D$, we eventually get

$$T(\omega_0, \xi_0) = (\hat{\eta}(\omega_0 \ln(\omega_f/\omega_0), \omega_0, \rho_0, \xi_0), \hat{\xi}(\omega_0 \ln(\omega_f/\omega_0), \omega_0, \rho_0, \xi_0)),$$

which has the desired regularity. $\square$

Remark 1.3
Notice that when $\omega_0 \to 0$, the Poincaré map goes in the invariant submanifold $\{\rho = 0\}$, although in infinite time.

Let us finally prove Proposition 1.4.

Proof of Proposition 1.4. We are going to use a generalization of the Poincaré-Dulac theorem. First we introduce some notation. Denote by H^l the space of homogeneous polynomials of degree l in $\mathbb{R}^n$ with smooth coefficients in $\xi \in \mathbb{R}^k$. We recall that for a linear vector field X that does not depends on ξ (and has no component in the ξ direction), $H^l = \mathrm{Im}[X, .]_{|H^l} + \ker[X, .]_{|H^l}$. A vector field Z is said to be resonant with X if $Z \in \ker[X, .]$.

Lemma 1.4
Let $X(x, \xi)$ be a smooth vector field in $\mathbb{R}^n \times \mathbb{R}^k$, $X(0, \xi) = 0$. Denote by X_1 its linear part. Then, if X_1 does not depend on ξ, there exist $g_i \in H^i \cap \ker[X_1, .]$, $i = 2, \ldots, l$ and a smooth vector field R_l with zero l-jet such that in a neighborhood of zero X is smoothly conjugate to

$$X_1 + g_2 + \cdots + g_l + R_l,\ \ l \in \mathbb{N}$$

Proof of Lemma 1.4. We will follow [55] and use induction on l, then treat the case $l = \infty$. For $l = 1$, the result is trivial: $X = X_1 + R_1$ where R_1 has zero first jet in zero. Suppose now that $g_1, \ldots, g_{l-1}$, and R_{l-1} are as in lemma 1.4, $l \geqslant 2$. R_{l-1} has a zero $l-1$ jet (in zero), and then can be written as $R_{l-1} = [X_1, Z] + g_l + R_l$, where $Z \in H^l$, $g_l \in \ker[X_1, .]$ and R_l is a smooth vector field with zero l-jet. $[X, Z] = [X_1, Z] + \sum_{i=2}^{l} [g_i, Z] + [R_{l-1}, Z] = [X_1, Z] + R'_l$, where R'_l has zero l-jet. Now, note ϕ_Z the flow of Z, and consider $(\phi_Z^t)_* X := X^t$. We have $\frac{d}{dt} X^t = [X, Z] = [X_1, Z] + R'_l$, so that $X^t = X^0 + t[X_1, Z] + R_{l,t}$, with $j^l(R_{l,t})(0) = 0$. Since Z is a homogeneous polynomial of degree l, it has zero $l-1$ jet, and X and X^t have the same $l-1$ jet, that means $X^0 = X_1 + g_2 + \cdots + g_l + [X_1, Z]$. Finally, we chose $t = -1$ to get $X^{-1} = X_1 + g_2 + \cdots + g_l + R_{l,-1}$, and ϕ_Z^{-1} conjugates the two vector field, which ends the proof of the finite case. The above construction provide a sequence of formal diffeomorphisms $\varphi_l = \phi_Z^{-1}$ ($Z \in H^l$) such that $(\varphi_l)_* X = X_1 + g_2 + \cdots + g_l + R_l$. Also notice that φ_l and φ_{l+1} have the same l-jet. This define a sequence of coefficient $g_l(\xi)$ for all l. □

Now, by that a generalization of Borel theorem, proved by Malgrange in [38], we know that there exists a smooth function ψ such that the l jet of φ_l and ψ are identical for all $l \in \mathbb{N}$. We can also realize, using the same theorem, the formal series given by the resonant monomials by a smooth vector field X^∞. Thus we have $\psi_*(X) = X^\infty + R_\infty$, where R_∞ has zero infinite jet. Thus, we begin by looking for monomials that are resonant with the linearized vector field of Y, $Y_1 = -\rho\frac{\partial}{\partial\rho} + s\frac{\partial}{\partial\omega}$ (monomials X for whom $[Y_1, X] = 0$). First notice that the Lie bracket with Y_1 treat ξ as a constant: The map $X \mapsto [Y_1, X]$ is linear in ξ. Such monomials can be written $a(\xi)\rho^i\omega^j\frac{\partial}{\partial\rho}$, $b(\xi)\rho^i\omega^j\frac{\partial}{\partial\omega}$, and $c(\xi)\rho^i\omega^j\frac{\partial}{\partial\xi}$. We get:

$$[Y_1, a(\xi)\rho^i s^j\frac{\partial}{\partial\rho}] = (i-j-1)a(\xi)\rho^i\omega^j\frac{\partial}{\partial\rho}$$

$$[Y_1, b(\xi)\rho^i\omega^j\frac{\partial}{\partial\omega}] = (i+1-j)b(\xi)\rho^i\omega^j\frac{\partial}{\partial\omega}$$

and

$$[Y_1, c(\xi)\rho^i\omega^j\frac{\partial}{\partial\xi}] = (i-j)c(\xi)\rho^i\omega^j\frac{\partial}{\partial\xi}.$$

The monomials we are looking for are thus: $a(\xi)\rho u^k\frac{\partial}{\partial\rho}$, $b(\xi)su^k\frac{\partial}{\partial\omega}$ and $c(\xi)u^k\frac{\partial}{\partial\xi}$, $k \in \mathbb{N}$. The lemma allow us to state that the infinite jet of Y can be formally developed on the resonant monomials, ie, Y is formally conjugate to

$$W : \begin{cases} \rho' = -\rho(1 + \sum_{i\geqslant 1} a_i(\xi)u^i) \\ \omega' = \omega(1 + \sum_{i\geqslant 1} b_i(\xi)u^i) \\ \xi' = \rho\sum_{i\geqslant 1} c_i(\xi)u^i \end{cases}.$$

It remains to realize those series by smooth functions. By the remark above, there exist Y^∞ a smooth vector field on O such that $W = Y^\infty + R_\infty$ where R_∞ is a smooth function with zero infinite jet along D. At this stage we have Y is smoothly equivalent to $Y^\infty + R_\infty$.

The last step consists in killing the flat perturbation R_∞. This can be achieved by the path's method: Instead of looking for a diffeomorphism sending $Y_0 := Y$ on $Y_1 := Y^\infty + R_\infty$, we search for a one parameter family (path) of diffeomorphism (g_t) such that

$$g_t^* Y_0 = Y_t, \tag{1.2.21}$$

Y_t being a path of vector fields joining Y_0 and Y_1. Consider the linear path $Y_t = (1-t)Y_0 + tY_1$, $t \in [0,1]$. Differentiating (1.2.21) with respect to t we get

$$\frac{\partial}{\partial t}(g_t^* Y_0) = \dot{Y}_t = Y_1 - Y_0 = R_\infty. \tag{1.2.22}$$

Now, the family g_t define a family of vector fields Z_t by $Z_t(g_t(x)) = \frac{\partial}{\partial t} g_t(x)|_t$, reciprocally by integrating Z_t we obtain the desired path of diffeomorphism. Thus (1.2.22) can be rewritten

$$[Y_t, Z_t] = R_\infty. \tag{1.2.23}$$

We just showed that getting rid of the flat perturbation R_∞ boils down to find a solution to the equation (1.2.23). It has been proved in [52], theorem 10, that this equation has a solution. □

1.2.4 Application to mechanical systems

In this section, we specify our study to mechanical systems coming from an autonomous potential, and go further in the particular case of orbit transfer problem with gravitational coplanar two or three body potential. Ie, we consider systems of the form

$$\ddot{q} + \nabla V(q) = 0, \tag{1.2.24}$$

where V is a smooth function defined on M.

Before applying the result of section 1.2.2, let us make the following important remark: In both cases, the distribution generated by F_1 and F_2 is involutive as the two vector fields actually commute:

$$[F_1, F_2] = 0. \tag{1.2.25}$$

Furthermore, we have:

Proposition 1.5
Assumption (A) holds for the Keplerian and circular restricted three-body problems.

This is actually the consequence of a more general statement (see [20] for the proof).

Lemma 1.5
A second order controlled system on $\mathbb{R}^m$

$$\ddot{q} + g(q, \dot{q}) = u$$

is a control-affine system on $\mathbb{R}^{2m}$ with an involutive distribution $\{F_1, \dots, F_m\}$ and a drift F_0 such that $\{F_1, \dots, F_m, F_{01}, \dots, F_{0m}\}$ has maximum rank.

Local properties.

The following proposition has been proved in [23]. It is a direct consequence of remark 1.2 and (1.2.25).

Proposition 1.6
The switching corresponds to instant rotation of angle π of the control (the so-called π-singularity). If t is a switching time, $u(t_-) = -u(t_+)$.

Proposition 1.6 immediately implies that the switching function $t \mapsto (H_1, H_2)(z(t))$ is $\mathcal{C}^1$. Now applying theorem 2.1, we obtain:

Proposition 1.7
Let $O_{\bar{z}}$ be a neighborhood of $\bar{z} \in \Sigma$: The local extremal flow of mechanical systems of the type (1.2.24) is piecewise smooth, and stratified into: $O_{\bar{z}} = S_0 \cup S^s \cup \Sigma$, S^s being the co-dimension one manifold of initial conditions leading to the singularity, and S_0 its complementary. The flow is smooth on each strata and continuous on $O_{\bar{z}}$:

- *if $z_0 \in S^s$ the extremal from z_0 has exactly one π-singularity in $O_{\bar{z}}$,*
- *Otherwise the extremal from z_0 has no π-singularity.*

Besides, the regular-singular transition is Lipschitz continuous.

The result apply to the controlled (RCTBP), but not in the more general elliptic case defined below.

Proof. Since (A) is checked, we just need to insure that $\Sigma = \Sigma_-$ in this case. However $H_{12} \equiv 0$ because of (1.2.25), implying $H_{01}^2(z) + H_{02}^2(z) > H_{12}(z)^2$ for all $z \in T^*M$. The last statement is a direct consequence of theorem 1.5. □

Global properties for the restricted three body problem.

Those switchings are the so-called π-singularities. Since there is no accumulation of switching points, there is only a finite number of such singularities on a time interval $[0, t_f]$. The results of this section apply to a more general problem than the (CRTBP), namely, we defined the elliptic restricted three body problem (ERTBP) in example 1.2. Let $q \in M = \mathbb{R}^2 \backslash \{-\mu, 1-\mu\}$ be the position vector, and note μ the mass ratio of the two primaries. Denote q^1 and q^2 the positions of the two primaries, which by definition, are in elliptic motion around their center of mass. We recall the dynamics

$$\ddot{q} + \nabla V_\mu(t, q) = u,$$

with $V_\mu(t,q) = \frac{1-\mu}{|q-q^1(t)|} + \frac{\mu}{|q-q^2(t)|}$. One can notice that when $\mu = 0$, we are in the Keplerian case. The non-autonomous Maximized Hamiltonian is the one given in example 1.2

$$H(t,z) = p_q.v - \nabla V_\mu(t,q).p_v + \|p_v\|, \text{ with } z = (x,p).$$

with of course p_q, (resp. p_v) being the adjoint coordinate of q, (resp. v). The controlled circular restricted three body problem is a reduction of the (CRTBP) where the two primaries are in circular motion around their center of mass. Its dynamics can be express as an autonomous system, in the rotating frame. Written on the convenient affine control system form: $x = (q,v)$,

$$\dot{x} = F_0(x) + u_1F_1(x) + u_2F_2(x)$$

with

$$F_0(x) = F_0(x) = v.\partial_q + (-q + (1-\mu)\frac{q+\mu}{|q+\mu|^3} + \mu\frac{q-1+\mu}{|q-1+\mu|^3} + 2Jv).\partial_v,$$

$F_1(x) = \partial_{v_1}$, $F_2(x) = \partial_{v_2}$ in Cartesian coordinates. The maximized Hamiltonian of the (CRTBP) given by the maximum principle is:

$$H(z) = p_q.v - J_\mu(q,u).p_v + |p_v|,$$

with

$$J_\mu(q,v) = -q + (1-\mu)\frac{q+\mu}{|q+\mu|^3} + \mu\frac{q-1+\mu}{|q-1+\mu|^3} + 2Jv.$$

We are now going to investigate global properties of such switching. Namely, the next proposition bounds the number of π-singularity along a time optimal trajectory.

Definition 1.5
We define $\delta = \inf_{[0,t_f]} |q(t)|$, $\delta_1 = \inf_{[0,t_f]} |q(t) - q^1(t)|$, $\delta_2 = \inf_{[0,t_f]} |q(t) - q^2(t)|$. This quantities represents the distance to the collisions in the two body, and restricted three-body problems respectively. Finally note $\delta_{12}(\mu) = \frac{\delta_1\delta_2}{((1-\mu)\delta_2^3 + \mu\delta_1^3)^{1/3}}$.

Estimate on the global number of switching can be obtain by Sturm like theorems, we denote by $[x]$ the integer part of a real number x.

Proposition 1.8 (Bound on switchings)

- *In the Keplerian case, there is at least a time interval of length $\pi\delta^{3/2}$ between two π-singularities. On a time interval $[0,t_f]$ the maximum amount of such singularities is $N_0 = [\frac{t_f}{\pi\delta^{3/2}}]$.*

- *In the controlled elliptic three-body problem with a mass ratio μ, there is at least a time interval of length $\pi\delta_{12}(\mu)^{3/2}$ between two π-singularities. On a time interval $[0, t_f]$ the maximum amount of such singularities is $N_\mu = [\frac{t_f}{\pi\delta_{12}(\mu)^{3/2}}]$.*

One can notice that $\delta_{12}(0) = \delta$, which makes the proposition coherent. To the author's knowledge, the only other bounds on the number of switchings were given in [23] with an extra hypothesis for the minimum time Kepler problem - namely, there can be only one switching if they occur at the perigee or apogee of the ellipsis - and, for linear systems, in the thesis of Carolina Biolo. The proof is a immediate consequence of the more general lemma:

Lemma 1.6
Let us consider the minimum-time control system coming from a $\mathcal{C}^2$ potential $V : \mathbb{R} \times O \to \mathbb{R}$, $O \subset \mathbb{R}^n$,

$$\ddot{q} + \nabla V_t(q) = u, \|u\| \leqslant 1. \tag{1.2.26}$$

Let $A_t(q) \in S_n(\mathbb{R})$ a continuous matrix, such that for all time t, and $q \in O$, $A_t(q) \geqslant \nabla^2 V_t(q)$, then the following statement holds: If $\bar{t}_1$ and $\bar{t}_2$ are switching times for (1.2.26), there exists a non trivial solution of $\ddot{y} + A_t(q(t))y = 0$ vanishing in $\bar{t}_1$ and $t' < \bar{t}_2$.

Proof of Lemma 1.6. Applying the P.M.P. to (1.2.26), one gets the maximized Hamiltonian $H^{\max}(q, v, p_q, p_v) = p_q.v - p_v.\nabla V_t(q) + |p_v|$, and the feedback $u^* = \frac{p_v}{|p_v|}$. It remains to study the zeros of the adjoint state p_v to have access to the switching times. The equation on p_v is a second order linear equation:

$$\ddot{p}_v + \nabla_q^2 V_t(q) p_v = 0. \tag{1.2.27}$$

The following Sturm-like theorem is due to Morse and will allow us to conclude (see [43]):

Theorem 1.6 (Morse)
Let a, $b \in \mathbb{R}$, with $b > a$. Consider the two linear second order equations

$$z'' + P(t)z = 0, \tag{1.2.28}$$

and

$$z'' + Q(t)z = 0, \tag{1.2.29}$$

with $P(t)$, $Q(t)$ be two symmetric continuous $n \times n$ matrices, such that $Q(t) - P(t) \geqslant 0$, and assume there exists a $\bar{t}$ with $Q(\bar{t}) - P(\bar{t}) > 0$. If (1.2.28) has a non trivial solution y, $y(a) = y(b) = 0$, then (1.2.29) has a non trivial solution which vanishes in a and $c < b$.

If $A_t(q)$ is a matrix as mentioned in Lemma 1.6, Morse's theorem gives the result by taking $Q(t) = A_t(q(t))$ and the lemma is proved. $\square$

Proof of Proposition 1.8. We will only be interested the second statement, since it implies the first one. Recall the dynamics of the third mass is:

$$\ddot{q} = -\nabla_q V_\mu(t,q) + u,$$

with $V_\mu(t,q) = -\frac{1-\mu}{|q-q^1(t)|} - \frac{\mu}{|q-q^2(t)|}$, the non-autonomous potential. Let

$$A_t(q) = \begin{pmatrix} 1 + \frac{1-\mu}{|q-q^1(t)|^3} + \frac{\mu}{|q-q^2(t)|^3} & 0 \\ 0 & \frac{1-\mu}{|q-q^1(t)|^3} + \frac{\mu}{|q-q^2(t)|^3} \end{pmatrix} \in \mathscr{M}_2(\mathbb{R}).$$

A straightforward calculation shows that

$$\det(A_t(q) - \nabla_q^2 V_\mu(t,q)) = 3\left[(1-\mu)\frac{(q_2 - q_2^1)^2}{|q-q^1|^5} + \mu\frac{(q_2 - q_2^2)^2}{|q-q^2|^5}\right] > 0$$

as long as we don't have $q_2(t) = q_2^1(t) = q_2^2(t)$. But this cannot happen on the whole trajectory, and so Morse's theorem apply. By our lemma above, the minimum-time interval between two switching times is greater than the time interval between conjugate times of the solutions of: $z_2'' + (\frac{1-\mu}{|q-q^1(t)|^3} + \frac{\mu}{|q-q^2(t)|^3})z_2 = 0$. But of course, $\frac{1-\mu}{|q-q^1(t)|^3} + \frac{\mu}{|q-q^2(t)|^3} \leqslant \frac{1-\mu}{\delta_1^3} + \frac{\mu}{\delta_2^3}$, and by Sturm's theorem in dimension one, solutions of this equation cannot have two conjugate times in a interval of length smaller than $\frac{\pi}{\sqrt{\frac{1-\mu}{\delta_1^3} + \frac{\mu}{\delta_2^3}}}$. Finally, the distance between two switching times is in fact greater than

$$\frac{\pi \delta_1^{3/2} \delta_2^{3/2}}{\sqrt{(1-\mu)\delta_2^3 + \mu\delta_1^3}} = \pi\delta_{12}(\mu)^{3/2}.$$

$\square$

Proposition 1.8 has an interesting dynamical consequence. The number of heteroclinic intersections between the stable and unstable foliations to the normally hyperbolic manifolds of he regularized system for the RC3BP is actually bounded by N_μ.

1.3 Conclusion and perspectives

We have described the singularities of the minimum time control of some affine control systems in a special case (Σ_-) meant to include mechanical systems. The other cases, which were not treated in this chapter, as well as a study of the whole picture, are the topic of the following one. One of the natural extensions for this work is to study the influence of averaging on the stratification. Indeed, in the controlled Kepler problem, one can average with respect to the fast angle, [20, 13] and reduce the dimension of the problem while regularizing the dynamics. Thus, the projection of the singular locus Σ becomes of codimension one and extremals will cross it generically. Fortunately, averaging regularizes the Hamiltonian and the study of the singularities still occurring has been initiated in the previously cited papers and should be pursued. We end this chapter with a remark: theorem 2.1, could be generalized even further. Indeed consider the general n-dimensional systems with m-dimensional control

$$\dot{x} = F_0(x) + \sum_{i=1}^{m} u_i F_i(x),$$

and make the Euclidean division of n by m: $n = km + l$, $k \geqslant 1$, $0 \leqslant l < m$. A similar proof, though a lot more technical, could be achieved with a similar assumption:

$$\mathbf{rank}(F_1(x), \ldots, F_m(x), \underbrace{\ldots}_{k \text{ times}}, F_{00\ldots01}(x), \ldots, F_{00\ldots0k}(x)) = n$$

for almost all $x \in M$, with the notations defined in section 1.

Chapter 2

Singularities of minimum time control-affine systems

Abstract

We provide an existence result for extremals with switchings for generic minimum time affine control systems with double input control on a 4 dimensional manifold. The nilpotent case is carefully studied through successive blow ups. We also give the regularity of the extremal flow around the singular locus and describe the jump on the control at a switching time.

Affine control systems generalize sub-Riemannian geometry, by adding a drift, and arise naturally from controlled mechanical systems. We handle the case of minimizing the final time, for an affine control system defined on a connected 4-dimensional manifold M under a generic assumption given below. The control u in dimension 2 (double input case), and contained in the unitary euclidean ball. In this case, the research of optimal trajectories lead, by Pontrjagin's Maximum Principle to study a singular Hamiltonian system defined on the cotangent bundle of the manifold M. We are interested in the local behavior of the extremal flow around the singularities. The singular locus which is of codimension two, is also called the switching set, where a switch - a discontinuity of the optimal control, is susceptible to occur. We compare it with single-input control systems, ie, when the control is scalar. More precisely it is of the form

$$\dot{x} = F_0(x) + uF_1(x),\ x \in M,\ F_i \in \Gamma(TM).$$

They have been extensively studied, and many things are known. In this chapter, we give a precise description of the behavior of the extremal flow around the singularities. We answer two important questions which remained open:

- Is there trajectory going to any point of the singular locus ?
- How does the flow behave when those singularities are crossed ?

Indeed, the singular locus is partitioned into three subsets, and what happened at the frontier was unknown. This case detailed below was more difficult to handle, being the frontier of bifurcation, and higher order analysis of the dynamics was necessary. Those questions have simpler answers when one consider the problem with the control set U being a box; which, in the single input case, makes no difference, obviously.

2.1 Setting

As in the last chapter, M is a 4-dimensional manifold, x_0, $x_f \in M$ and consider the following time optimal control system:

$$\begin{cases} \dot{x}(t) = F_0(x(t)) + u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)), \quad t \in [0, t_f], \quad u \in U \\ x(0) = x_0 \\ x(t_f) = x_f \\ t_f \to \min \end{cases} \tag{2.1.1}$$

where the F_i's are smooth vector fields. Denoting $F_{ij} := [F_i, F_j]$, as in the previous chapter we make the following generic assumption on the distribution $\mathcal{D} = \{F_0, F_1, F_2\}$:

$$\det(F_1(x), F_2(x), F_{01}(x), F_{02}(x)) \neq 0, \text{ for almost all } x \in M. \tag{A}$$

This is slightly stronger than the assumption that the distribution $\mathcal{D}$ is of step 2, but it is natural in the applications to mechanical systems, for instance. By Pontrjagin's Maximum Principle, optimal trajectories are the projection of integral curves of the maximized Hamiltonian system defined on the cotangent bundle of M by

$$H^{\max}(z) := H_0(z) + \sqrt{H_1^2(z) + H_2^2(z)}, \; z \in T^*M$$

where we have denoted $H_i(z) := \langle p, F_i(x)\rangle$ in canonical coordinate $z = (x, p) \in T_x^*M$. See [22] or [4] for more details about Pontrjagin's Maximum Principle. It also implies the feedback control

$$u = \frac{1}{\sqrt{H_1^2 + H_2^2}}(H_1, H_2)$$

whenever $(H_1, H_2) \neq (0, 0)$. The minimization of the final time implies $p(t) \neq 0$ for all $t \in [0, t_f]$. The extremal system is smooth outside the singular locus, or switching surface,

$$\Sigma := \{H_1 = H_2 = 0\}.$$

Definition 2.1 (Bang and bang-bang and singular extremals)
An extremal $z(t)$ is said to be bang *if $(H_1, H_2)(z(t)) \neq (0, 0)$ for all t. It is* bang-bang *if it is a concatenation of bang arcs. We say it is* singular *if it is contained in the singular locus.*

For single input systems, if one consider bounded controls, taking values in $[-1, 1]$, then the maximized Hamiltonian is

$$H(x, p) = H_0(x, p) + |H_1(x, p)|$$

with as before $H_i(x, p) = \langle p, F_i(x)\rangle$, $i = 0, 1$, and $u = \text{sign}(H_1)$ when H_1 is nonzero. The singular locus for this problem is obviously $\{H_1 = 0\}$. The singular controls can then be easily calculated by differentiating the relation $H_1(z(t)) = 0$. For the so-called oder one singular extremal, meaning, singular extremals z such that $H_{101}(z(t)) \neq 0$, we have

Proposition 2.1
The singular flow for single input systems is given by the Hamiltonian $H^s = H_0 - \frac{H_{100}}{H_{101}} H_1$, *with the singular control is given by* $u^s = \frac{\{\{H_1,H_0\},H_0\}}{\{\{H_1,H_0\},H_1\}}$.

Discontinuities of the control u along an extremal are called switchings, a time $\bar{t}$ at which a switch occurs is called switching time, and $z(t)$, a switching point. Bang extremals are the one that do not cross Σ. Now we tackle the double-input system. A singular minimum time extremal is such that $H_{12}(z(t)) \neq 0$, see remark 1.2 below. One can also see [20].

Proposition 2.2
There exists a singular flow inside Σ, *on which we have the control feedback:* $\tilde{u}_s = \frac{1}{H_{12}}(-H_{02}, H_{01})$, *and the singular flow is smooth. It is solution of the Hamiltonian system given by* $\tilde{H}^s = H_0 - \frac{H_{02}}{H_{12}} H_1 + \frac{H_{01}}{H_{12}} H_2$.

Proof. The proposition is obtained by differentiating the identically zero switching function $(H_1, H_2)(z(t))$ with respect to the time. □

From [20] and [22], we know Σ is partitioned into three subsets, leading to three very different local dynamics in their neighborhoods, namely (we use the notation $H_{ij} = \{H_i, H_j\}$)

$$\begin{aligned}\Sigma_- &= \{H_{12}^2(z) < H_{02}^2(z) + H_{01}^2(z)\}\\ \Sigma_+ &= \{H_{12}^2(z) > H_{02}^2(z) + H_{01}^2(z)\}\\ \Sigma_0 &= \{H_{12}^2(z) = H_{02}^2(z) + H_{01}^2(z)\}.\end{aligned}$$

The behavior of the flow in a neighborhood of Σ_0 remains open, as the case Σ_- (and even Σ_+ in a slightly different setting) were settled in [22, 2], but we attempt to provide in the next section a unification of the different viewpoints.

Remark 2.1
Note that, since along a Pontrjagin extremal, the adjoint state p *cannot vanish,* (A) *is equivalent to*

$$H_1^2 + H_2^2 + H_{01}^2 + H_{02}^2 > 0.$$

Let us take a look to the single input case: the study of switchings is very simplified by te following fact: under generic hypothesis, one can define switching times by the implicit function theorem for order one switchs (switching occur when H_1 vanishes, and the Hamiltonian vector field $X_{H_0} + X_{H_1}$ is well defined). This fact remain true when U is a box instead of a ball, see [48]. The components of the control u just go from $+1$ to -1 or the opposite.

2.2 The bifurcation formulation

In this section we introduce a parameter in the previous dynamical system in order to unify the viewpoints. Thanks to (A), one can make the change of coordinates:

$$z = (x, p) \in T^*M \mapsto (x, H_1, H_2, H_{01}, H_{02}) \in M \times \mathbb{R}^4.$$

Then use a polar blow up by setting $(H_1, H_2) = \rho(\cos\theta, \sin\theta)$ and $(H_{01}, H_{02}) = r(\cos\phi, \sin\phi)$. The dynamics boils down to the system:

$$\begin{cases} \dot\rho = r\cos(\theta - \phi) \\ \dot\theta = \frac{1}{\rho}(H_{12} - r\sin(\theta - \phi)) \\ \dot\xi = h(\rho, \theta, \xi) \end{cases} \tag{2.2.1}$$

where $\xi = (x, r, \phi)$ and h is a smooth function defined on an open set O of $\mathbb{R}\times\mathbb{R}\times D$, D being a compact domain of $\mathbb{R}^6$; h has values in $\mathbb{R}^6$. We set $\psi = \theta - \phi$, and rescale the time according to $ds_1 = rdt$: by remark 2.1, r is never 0 in a neighborhood of Σ, meaning this defines a diffeomorphism and new dynamics is conjugate the one of system 2.2.1. This boils down to study a general system with the following structure (the derivation still being noted "$\cdot$"):

$$\begin{cases} \dot\rho = \cos\psi \\ \dot\psi = \frac{1}{\rho}(g(\rho, \psi, \xi) - \sin\psi) \\ \dot\xi = \tilde h(\rho, \psi, \xi) \end{cases} \tag{2.2.2}$$

where g, $\tilde h$ are smooth functions defined on an open set O of $\mathbb{R}\times\mathbb{R}\times D$, D being a compact domain of $\mathbb{R}^k$; g depends smoothly on $\rho(\cos\theta, \sin\theta)$. One can set $s = \psi - \pi/2$, besides, since ξ is constant when $\rho = 0$, we can assume without loss of generality $\bar\xi = 0$, and study the behavior of this system around the equilibria $(0, 0, 0)$. By a small abuse of the notations, we still note $g(\rho, s, \xi) = g(\rho, s + \pi/2, \xi)$, and we have $g(\rho, s, \xi) = a(\xi) + O(\rho)$ near $\rho = 0$.

The three cases Σ_+, Σ_- and Σ_0 correspond to $g(0, 0, 0) > 1$, $g(0, 0, 0) < 1$ and $g(0, 0, 0) = 1$, see [22] for more details. We set $a(\xi) = 1 + \alpha + a_0(\xi)$, with $a_0(0) = 0$. So that the bifurcation parameter giving the three cases is $\alpha = a(0) - 1$ and

$$g(\rho, s, \xi) = 1 + \alpha + a_0(\xi) + O(\rho).$$

Then (2.2.2) becomes (with slight abuse of notation):

$$(Z) : \begin{cases} \dot\rho = -\sin s \\ \dot s = \frac{1}{\rho}(1 + \alpha + a_0(\xi) - \cos s + O(\rho)) = \frac{G_\alpha(\rho, s, \xi)}{\rho} \\ \dot\xi = \tilde h(\rho, s, \xi) \end{cases} \tag{2.2.3}$$

2.2.1 The case Σ_-.

In the system above, the equilibrium $0 \in \Sigma_-$ if and only if $\alpha < 0$. We recall that for the sake of clarity, we will work in a neighborhood of 0, keeping in mind that we don't lose generality (and the results holds for any point). That case was settled in chapter I by theorem 2.1 recalled below:

Theorem 2.1
In a neighborhood $O_{\bar{z}}$ with $\bar{z} \in \Sigma_-$, existence and uniqueness hold, all extremal are bang-bang, with at most one switch. The extremal flow $z : (t, z_0) \in [0, t_f] \times O_{\bar{z}} \mapsto z(t, z_0) \in M$ is piecewise smooth. More precisely, it can be stratified as follows:

$$O_{\bar{z}} = S_0 \cup S^s \cup S^u \cup \Sigma$$

where S^s (resp. S^u) is the codimension-one submanifold of initial conditions leading to the switching surface (resp. in negative times), $S_0 = O_{\bar{z}} \backslash (S^s \cup \Sigma)$. Both are stable by the flow, which is smooth on $[0, t_f] \times S_0$, and on $[0, t_f] \times S^s \backslash \Delta$ where $\Delta = \{(\bar{t}(z_0), z_0),\ z_0 \in S^s\}$, and $\bar{t}(z_0)$ is the switching time of the extremal initializing at z_0, and continuous on $O_{\bar{z}}$.

In [22], the authors also studied the regular-singular transition between the strata, and exhibited log-type singularities.

Remark 2.2
By proposition 2.2, there is no singular flow contained in Σ_-, otherwise $\|u_s\|^2 = \frac{H_{02}^2 + H_{01}^2}{H_{12}^2} > 1$ which violates our constraint on u.

2.2.2 The case Σ_+.

This corresponds to $\alpha > 0$. We prove the following, see also [2], theorem 3.5:

Proposition 2.3
In a neighborhood of a point $\bar{z}$ in Σ_+, there is no switch, and the extremal flow is smooth, i.e., Σ_+ is never crossed. In other words, ρ does not vanish in (2.2.3).

Proof. By the analysis above, this boils down to prove that, if $\alpha > 0$, along an extremal z, ρ never vanishes in a (relatively compact) neighborhood $\bar{O}$ of $(0, 0, 0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k$. But in such a neighborhood, the differential of u is bounded by below by a negative constant $-a < 0$. Now recall $\rho \dot{s} = 1 + \alpha + u(\rho, s, \xi) - \cos s := \Theta$ so that

$$\frac{d}{dt}(\rho\Theta) = \dot{\rho}(1 + \alpha - \cos s + u(z)) + \rho(\sin s \dot{s} + du(z).\dot{z}) = \rho du(z).Z(z)$$

We get $\frac{d}{dt}(\rho\Theta) > -a\rho = \rho\Theta(-a/\Theta)$. Eventually, since $\alpha > 0$ and $u(0,0,0) = 0$, if $\bar{O}$ is small enough, there exists two positive constant K, k with $K > \Theta > k > 0$. So that, along an extremal

$$\frac{d}{dt}(\rho\Theta) > -\frac{a}{k}\rho\Theta.$$

By integration on a arbitrary time interval $[0,t]$, we end up with

$$\rho(t) > \frac{\rho_0\Theta_0}{K}e^{-\frac{a}{k}t},$$

and the proposition follows. $\square$

Despite the absence of switch, there exists a singular flow inside Σ_+, as shown in [22], however, singular extremal lying in Σ_+ cannot be optimal by the Goh condition, [20].

2.2.3 The bifurcation $\alpha = 0$: case Σ_0.

This is the main topic of this chapter. In this case, the two equilibria considered in Σ_- merge, and we obtain one nilpotent equilibrium that needs desingularization. Nevertheless, we will prove

Theorem 2.2
For generic systems (2.1.1), if assumption (A) holds. Let $\bar{z}$ be in Σ_0: there exists a unique trajectory passing through $\bar{z}$, either going in or going out of Σ_0 at $\bar{z}$. In the first case, this trajectory is connected to the singular flow in Σ_0.

This result contradicts the last part of theorem 3.5, in [2], a counter example in a particular case was given in [9] (nilpotent case). The next figure is a scheme of the whole behavior.

The regularity of the extremal flow is of theoretical and numerical importance, and we have:

Theorem 2.3
In a neighborhood $O_{\bar{z}}$ of a point $\bar{z} \in \Sigma_0$, the flow is well defined, continuous, and piecewise smooth. More precisely, there exists a stratification:

$$O_{\bar{z}} = S_0 \cup S^s \cup S_0^s$$

where

- *S_0^s is the submanifold of codimension 2 of initial conditions leading to Σ_0,*

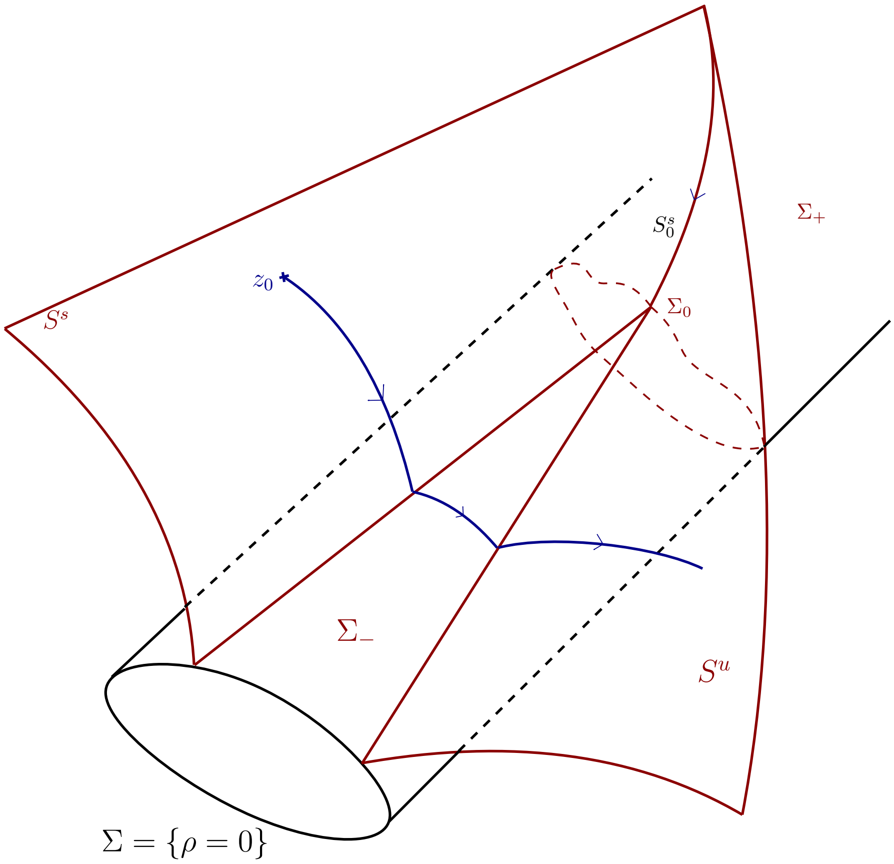

Figure 2.1: The stable and unstable manifold of Σ_- merging on Σ_0.

- *S^s (resp. S^u) is the submanifold of codimension 1 of initial conditions leading to Σ_-,*
- *$S_0 = O_{\bar{z}} \backslash (S_0^s \cup S^s)$.*

The extremal flow $z_{|S_0 \times [0,t_f]}$, $z_{|S^s \times [0,t_f] \backslash \Delta}$, $z_{|S_0^s \times [0,t_f] \backslash \Delta^0}$ *is smooth, and* $\Delta^0 = \{(\bar{t}(z_0), z_0),\ z_0 \in S_0^s\}$.

Namely, the previous theorem states that the extremal flow is smooth restricted to each strata, except, obviously, at the point and time where the singular locus is crossed (at those points, it is only Lipschitz in time alone). In the process we also obtain the kind of jump occurring on the extremal control, a switching on the control is called a π-singularity if it is a instant rotation of angle π, see [22].

Proposition 2.4
Consider the extremal $z(t)$ entering the singular locus in $z(\bar{t}) = \bar{z} \in \Sigma_0$,

- *If* $H_{12}(\bar{z}) = r(\bar{z})$, *the extremal control is continuous on* $[0, t_f]$,

- *If* $H_{12} = -r(\bar{z})$, *the extremal control has a* π*-singularity at time* $\bar{t}$.

Proof of theorem 2.2

To that end, we give a precise description of the behavior of the flow around such an equilibrium. Make the following change of time $ds_1 = \rho ds_2$ to regularize the vector field Z, we have $g(0,0,0) = a(0) = 1$, and end up with:

$$\begin{cases} \dot{\rho} = -\rho \sin s \\ \dot{s} = 1 + a_0(\xi) - \cos s + O(\rho) \\ \dot{\xi} = \rho \tilde{h}(\rho, s, \xi) \end{cases} \tag{2.2.4}$$

We make the following generic assumption:

$$da(0).\tilde{h}(0,0,0) \neq 0 \tag{2.2.5}$$

Then, we can order the coordinates of $\xi = (\xi_1, \xi_2 \dots, \xi_k)$ such that $\frac{\partial a}{\partial \xi_1}(0) \neq 0$. This implies $\Gamma = \{G_0(0, s, \xi) = 0\}$ is a dimension k manifold around $(s, \xi) = (0, 0) \in S^1 \times \mathbb{R}^k$. We can then chose coordinates $\tilde{\xi} = (\tilde{\xi}_1, \dots, \tilde{\xi}_k) = \Phi(\xi)$ such that $\tilde{\xi}_1 = a_0(\xi)$ meaning, $\Gamma = \{\tilde{\xi}_1 + 1 - \cos s = 0\}$. Then, set $\zeta = \tilde{\xi}_1$ to simplify the notations. Denote $' = \frac{\mathrm{d}}{\mathrm{d}s_2}$ to get

$$\begin{cases} \rho' = -\rho s + O(\rho s^3) \\ s' = \zeta + s^2/2 + O(\rho) + O(|s|^4) \\ \zeta' = c\rho + \rho O(\rho + |s| + |\tilde{\xi}|). \end{cases} \tag{2.2.6}$$

We do not write the derivative of the other component of $\tilde{\xi}$ as they do not influence the dynamics of (ρ, s). Besides, as we will exhibit below, only the first order terms in the derivative of ζ are relevant for the local dynamics around 0. In the previous equation, $c = \tilde{h}(0,0,0)_1$, so that assumption 2.2.5 prevent it from being 0. It will be clear from what follows that the terms of higher order are useless for the local analysis.

Blow up. To study the nilpotent equilibrium $(\rho, s, \zeta) = (0,0,0)$, we will use a specific blow-up process, called quasi homogeneous blow-up, see [27] chap. 1 for instance:

$$\begin{cases} \rho = u^3\bar{\rho} \\ s = u\bar{s} \\ \zeta = u^2\bar{\zeta} \end{cases}$$

with $(\bar{\rho}, \bar{s}, \bar{\zeta}) \in S^2_+$ the hemisphere $\rho \geqslant 0$, $u \in \mathbb{R}_+$. We will study the dynamics in the two following charts:

(i) For the interior of S^2_+, $\bar{\rho} = 1$, $(\bar{s}, \bar{\zeta})$ in a disc D^2, $u \geqslant 0$ in a neighborhood of the singularity $u = 0$.

(ii) For the the boundary of S^2_+, $(\bar{s}, \bar{\zeta}) \in S^1$, in a neighborhood of $(\bar{\rho}, u) = (0, 0)$.

The chart (i). Let us write the dynamics in the blown up coordinates

$$\varphi(\rho, s, \zeta) = (u^3, u\bar{s}, u^2\bar{\zeta}).$$

The blown up vector field $\bar{X} = \frac{1}{u}\varphi_* X$ writes

$$\bar{X} : \begin{cases} u' = -\frac{1}{3}u\bar{s} + O(u^2) \\ \bar{s}' = \frac{5}{6}\bar{s}^2 + \bar{\zeta} + O(u) \\ \bar{\zeta}' = \frac{2}{3}\bar{s}\bar{\zeta} + c + O(u). \end{cases} \tag{2.2.7}$$

Thus, there is a unique equilibrium depending on c, which is solution of

$$\begin{cases} u = 0 \\ \bar{\xi} + \frac{5}{6}\bar{s}^2 = 0 \\ \frac{2}{3}\bar{s}\bar{\zeta} + c = 0 \end{cases} \tag{2.2.8}$$

ie, $m_0 = (u, \bar{s}_0, \bar{\zeta}_0) = (0, (\frac{9}{5}c)^{1/3}, -\frac{5}{6}(\frac{9}{5}c)^{2/3})$. The Jacobian matrix of $\bar{X}$ at m_0 is $\begin{pmatrix} -\frac{1}{3}\bar{s}_0 & 0 & 0 \\ * & \frac{5}{3}\bar{s}_0 & 1 \\ * & \frac{2}{3}\bar{\zeta}_0 & \frac{2}{3}\bar{s}_0 \end{pmatrix}$, giving the eigenvalue $-\frac{1}{3}\bar{s}_0$ in the direction of u. On $\{u = 0\}$ we get the two conjugate eigenvalues $\bar{s}_0(\frac{7}{6} \pm \frac{\sqrt{11}}{6}i)$. Thus m_0 is an hyperbolic equilibrium point, if $c > 0$ (implying $\bar{s}_0 > 0$), it has one dimension stable manifold, and a two dimensional unstable one. If $c < 0$, the situation is symmetric.

THE CHART (ii). Along ∂S^2_+ we set $\begin{cases} \bar\zeta = \cos\omega \\ \bar s = \sin\omega \end{cases}$ and proceed to the blow up $\rho = u^3\bar\rho,\ s = u\sin\omega,\ \zeta = u^2\cos\omega$. The pulled-back dynamics is

$$Y : \begin{cases} u' = \frac{u}{1+\cos^2\omega}(\sin\omega(\cos\omega + \frac{1}{2}\sin^2\omega) + c\bar\rho\cos\omega) + O(u^2) \\ \omega' = \frac{1}{1+\cos^2\omega}(\cos\omega(2\cos\omega + \sin^2\omega) - c\bar\rho\sin\omega) + O(\bar\rho u) \\ \bar\rho' = -\frac{\bar\rho}{1+\cos^2\omega}(\sin\omega(1+\cos^2\omega + \cos\omega + 1/2\sin^2\omega) + c\bar\rho\cos\omega) + O(\bar\rho u) \end{cases} \tag{2.2.9}$$

and is equivalent to

$$\bar Y : \begin{cases} u' = u(\sin\omega(\cos\omega + \frac{1}{2}\sin^2\omega) + c\bar\rho\cos\omega) + O(u^2) \\ \omega' = \cos\omega(2\cos\omega + \sin^2\omega) - c\bar\rho\sin\omega + O(\bar\rho u) \\ \bar\rho' = -\bar\rho(\sin\omega(1+\cos^2\omega + \cos\omega + 1/2\sin^2\omega) + c\bar\rho\cos\omega) + O(\bar\rho u) \end{cases} \tag{2.2.10}$$

In restriction to $\{\bar\rho = 0\}$, we obtain 4 equilibrium points, namely, the solutions of $\cos\omega(\sin^2\omega + 2\cos\omega) = 0$. In addition to the trivial $\pm\pi/2$, we end up with $\cos\omega = 1-\sqrt{2}$, this last equation gives two $\omega_0 \in]\pi/2,\pi[$ and $-\omega_0$. All this zeros are simple (in the direction of ω) so the dynamics on ∂S^2_+ can be deduced by the sign $\omega'(0) > 0$. Actually, from $\omega_{1,2}$ we get two lines of zero in the plane $\{\bar\rho = 0\}$, corresponding to the blow up of the parabola $\zeta = -s^2/2$.

Let us write the Jacobian matrix of $\bar Y$ in the plane $\bar\rho = 0$:

$$J = \begin{pmatrix} U(\omega) & * & * \\ 0 & \Omega(\omega) & * \\ 0 & 0 & R(\omega) \end{pmatrix}$$

with $U(\omega) = \frac{\sin\omega}{2}(2\cos\omega + \sin^2\omega)$, $\Omega(\omega) = -\sin\omega(2\cos\omega + \sin^2\omega - 2\cos^2\omega + 2\cos\omega$, and $R(\omega) = -\sin\omega(\cos\omega + \frac{3}{2}\sin^2\omega + 2\cos^2\omega)$. We still need two informations:

- the eigenvalues of $\pm\pi/2$ in the radial direction, given by $U(\pm\pi/2) = \mp 1$.
- the eigenvalues of the 4 equilibria in the direction of $\bar\rho$, given by $R(\pm\pi/2) = \mp\frac{3}{2}$ and $R(\omega_0) < 0$, $R(-\omega_0) > 0$. Now we have a clear description of the phase portrait in a neighborhood of ∂S^2_+ in Figure 2.2. $\pm\pi/2$ are hyperbolic equilibria and $\pm\omega_0$ are hyperbolic in restriction to S^2_+ (but not in dimension 3). The dynamics of (2.2.6) is also stable by perturbation by higher order terms.

GLOBAL DYNAMICS. We are now going to glue the studies in both charts to obtain the phase portrait on a whole neighborhood of the hemisphere. The main

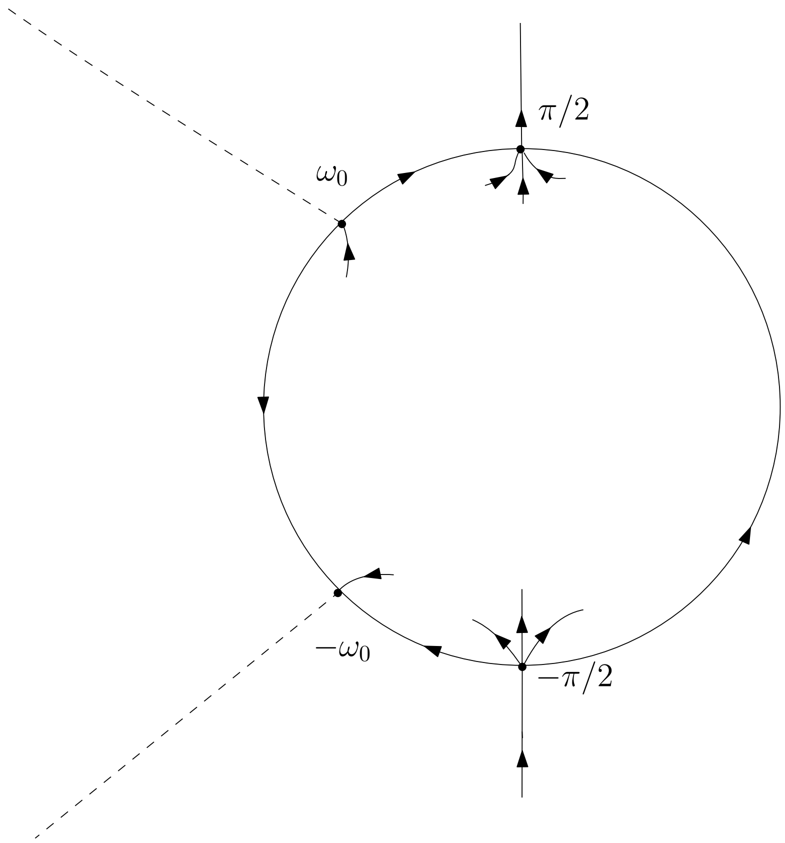

Figure 2.2: Phase portrait around ∂S^2_+

tool in that regard will be the following celebrated theorem from Poincaré and Bendixson, [11].

Theorem 2.4 (Poincaré-Bendixson)
Let X be a vector field in the plane, any maximal solution of $\dot{x} = X(x)$ contained in a compact set, is either converging to an equilibrium point or a limit cycle.

The equilibria of the flow restricted to S^2_+ (ie, $u = 0$) are as followed: $\pi/2 \in S^1 \cong \partial S^2_+$ is a stable node, likewise, $-\pi/2$ is an unstable node. The equilibrium m_0 is also unstable. ω_0, however, $-\omega_0$ has a one dimensional unstable manifold, which is going to be a separatrix. Its unidimensional unstable manifold is on ∂S^2_+. Besides, for ω_0, the opposite happens: It has a one dimensional stable manifold, and its unstable manifold is along ∂S^2_+. Now we will prove that $\bar{X}$ does not have any periodic orbits. This, according to Poincaré-Bendixson, will allow us to link the trajectory coming from unstable directions to the stable manifolds belonging to other singular points in S^2_+.

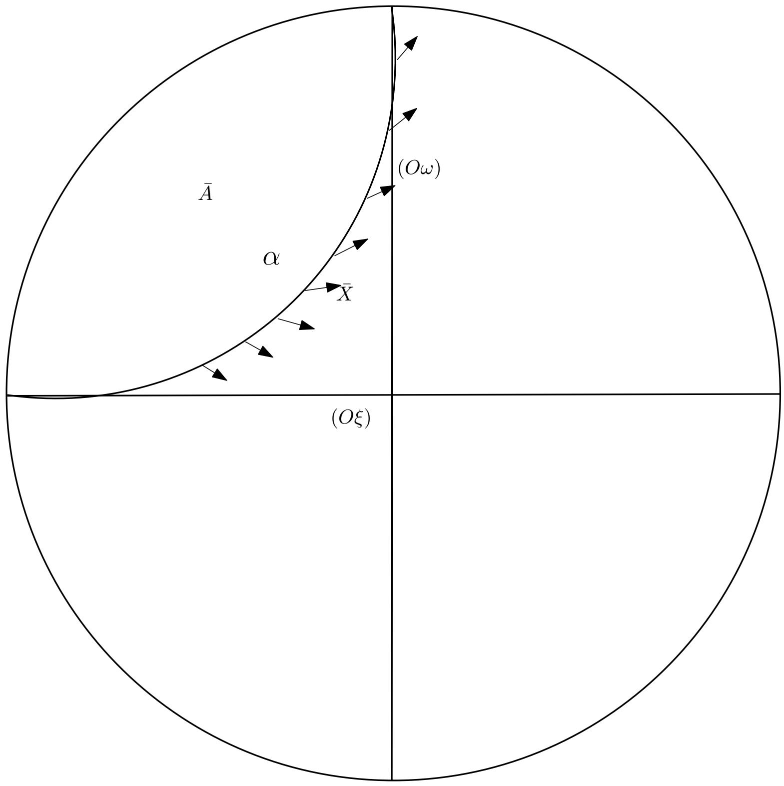

Figure 2.3: Building $\bar{A}$

Lemma 2.1
$\bar{X}$ does not have a periodic orbit on S^2_+.

Proof. Between chart (i) and (ii), we have the following change of charts :

$$\begin{cases} \bar{s} = \frac{\sin\omega}{\bar{\rho}^{1/3}} \\ \bar{\zeta} = \frac{\cos\omega}{\bar{\rho}^{2/3}}. \end{cases}$$

We can now define the two orthogonal axis $(O\bar{\zeta})$ and $(O\bar{s})$ in S^2_+, ∂S^2_+ included. In the chart (ii), $(O\bar{\zeta})$ is going from $\omega = \pi$ to $\omega = 0$. Consider the convex domain A, such that $\partial A = (O\bar{s})_+ \cup]\pi/2, \pi[\cup (O\bar{\zeta})_-$, then $m_0 \in \mathrm{Int}A$ is the only equilibrium of $\bar{X}$ in $S = \mathrm{Int}S^2_+$. The field $\bar{X}$ is positively collinear to $(0\bar{\zeta})_+$ in $(0,0)$, and transverse to those axis everywhere else. Thus, we can smooth the boundary of A corresponding to the part $(O\bar{\zeta})_- \cup (O\bar{s})_+$ by a curve α in order to make $\bar{X}$ transverse to ∂A. See figure 2.3

Denote $\bar{A}$ the part of S^2_+ such that $\bar{A} \subset A$ and $\partial\bar{A} =]\pi/2; \pi[\cup \alpha$. Now X is transverse to $\bar{A}$ and pointing outside $\bar{A}$. In $\bar{A}$, we have $\mathrm{div}(X) = 2\bar{s} > 0$. Now assume γ is a periodic orbit of $\bar{X}$. By Jordan's theorem, γ is the boundary of a compact set $D \subset S$. The result is then a consequence of the Poincaré-Hopf formula:

Theorem 2.5 (Poincaré-Hopf)
Let M be a compact manifold, and X a vector field that has isolated zeros on M. Then $\sum_{i=1}^{m} Index(x_i) = \chi(M)$, where the x_i are all the zeros of X in M, and χ denotes the Euler characteristic.

D being contractile, $\chi(D) = 1$, hence D contains at least one equilibrium point, and since m_0 is the only one in S, $m_0 \in D$. As a result, either γ lies in $\bar{A}$ or intersects α. Let us consider the first alternative: $\gamma \subset \bar{A}$. We have

$$0 < \int_D \operatorname{div}(\bar{X}) \mathrm{d}\bar{\zeta} \wedge \mathrm{d}\bar{s} = \int_D d(i_{\bar{X}}(\mathrm{d}\bar{\zeta} \wedge \mathrm{d}\bar{s})) = \int_\gamma i_{\bar{X}}(\mathrm{d}\bar{\zeta} \wedge \mathrm{d}\bar{s}) = 0$$

by Stokes formula, which excludes that case. Now, note that all intersection points between γ and α are transverse, since $\bar{X}$ is transverse to α: Thus, there is no tangency, and γ intersects α twice. But this is also excluded because $\bar{X}$ is only pointing outside $\bar{A}$. $\square$

Let us now exhibit the existence of separatrix. By the Poincaré-Bendixson theorem above, since there is no periodic orbits, in $\mathrm{Int}\bar{A}$ every trajectory converges to m_0 when the time tends to $-\infty$. $\omega_0 \in \partial\bar{A}$ has a stable manifold of dimension one, and this stable manifold lies inside $\mathrm{Int}\bar{A}$ (at least close to ω_0). This implies that the stable manifold from ω_0 converges to m_0 in negative infinite time. Apart from the equilibrium $\pi/2$, it is the only stable direction in S. That means all the other trajectories converge to the stable node (restricted to S^2_+) $\pi/2$, leading to the phase portrait of figure 2.4.

Back to the original system The initial problem is in dimension $k+2$ ($k = 6$ in our motivating control affine problem). From (2.2.4), we see that when $\rho = 0$, the ξ' vanish. Thus, in the blown up coordinates, when $u = 0$ (on S^2_+) or when $\bar{\rho} = 0$, the spaces $\{\tilde{\xi}_2 = \text{const}, \dots, \tilde{\xi}_k = \text{const}\}$ are preserved. Let us write their dynamics in the chart (i) (with obvious notation with respect to (2.2.4):

$$\begin{cases} \tilde{\xi}_2' = u^2 \tilde{h}_2(u, \bar{\rho}, \bar{s}, \tilde{\xi}) \\ \vdots \\ \tilde{\xi}_k' = u^2 \tilde{h}_k(u, \bar{\rho}, \bar{s}, \tilde{\xi}). \end{cases}$$

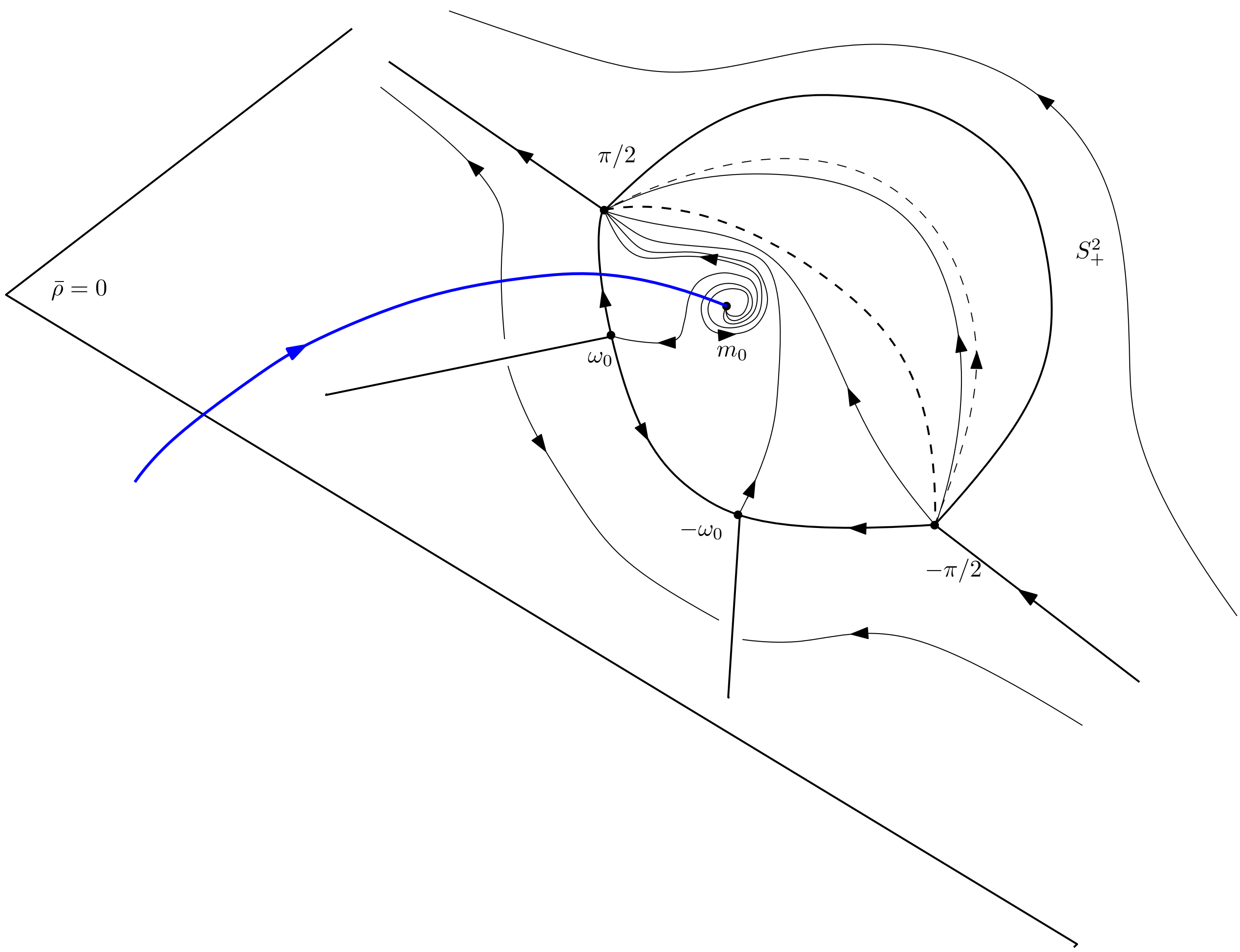

Figure 2.4: Phase portrait around S^2_+

The blown up space is $S^2_+ \times \mathbb{R}^{k-1}$, and the linear part of the total dynamics is the same as in (2.2.7), completed with zeros to obtain a $k+2$ matrix. As a result, in the initial system, the hyperbolic equilibrium point m_0 is replaced by a $k-1$ manifold of equilibria, denoted N, each having a stable one dimensional manifold in the direction of u, when $c > 0$, (resp. unstable hen $c < 0$): there exists a trajectory of (2.2.4) converging to each of this points. The stable manifolds of m_0 will allow us to prove theorem 2.2.

It remains to show that the trajectory coming from the stable manifold to m_0 is actually going to m_0 in finite time, for the original time t. We have been doing the following changes of times: $dt = \rho ds_1$, $ds_2 = r ds_1$, $ds_3 = u ds_2$ (ds_3 is the time in which we study the blown up system (2.2.7)), so that $dt = \frac{\rho}{ru} ds_3$. We will show that the interval of time from a point of the stable manifold to m_0 is finite. According to remark A, when $\rho = 0 \Rightarrow r > 0$, so that in a neighborhood O of Σ $r > 0$. Then in O, r is bounded below and above by two positive constant $A > r > B > 0$. In the blown up coordinates, $\rho = u^3 \bar{\rho}$, so that the previously mentioned interval of time is

$$\Delta t = \int_{s_3^0}^{+\infty} \frac{u^2(s_3)\bar{\rho}}{r(s_3)} ds_3 < \frac{1}{B} \int_{s_3^0}^{+\infty} u^2(s_3)\bar{\rho}(s_3) ds_3.$$

Notice that $\bar{\rho}$ is bounded by above by a positive constant K along the trajectory in the stable manifold, since it converges to m_0.

Now, the first line of system (2.2.7) is $u' = -\frac{1}{3} u \bar{s}$. Since $m_0 \in \{\bar{s} > 0\}$, if O is small enough, $u' < -cu$ for a constant $c > 0$. Then as along the stable manifold to m_0, we have $u(s_3) < u_0 e^{-cs_3}$ by integration between a time s_3 and s_3^0. So that finally,

$$\Delta t < K/B \int_{s_3^0}^{+\infty} u_0^2 e^{-2cs_3} ds_3 < +\infty.$$

So $\bar{z}$ is reached in finite time. From figure 2.4, and the fact that the $\tilde{x}_i$'s, $i > 1$ are constant on S^2_+ and when $\bar{\rho}$ vanishes, one can make the same time estimates to prove that the extremal goes out of S^2_+ in finite time, and as such is connected to the singular flow.

Proof of theorem 2.3

In the process of proving theorem 2.2, we obtained a clear description of the singular flow around a point of Σ_0. We will make the proof when $c > 0$, the opposite case being similar. The manifold $N_0 = \{m_0\} \times \mathbb{R}^{k-1} \cap O_{\bar{z}}$ in the blown up space is normally hyperbolic since m_0 is hyperbolic and each point is an equilibrium. In the direction of u, there is at one dimensional stable manifold at each of those points. Thus, as in [22], we can define the global stable manifold $S_1^0 = \bigcup_{z \in N_0} W^s(z)$, and from [33], it is $\mathcal{C}^\infty$-smooth. The construction of the strata S^s is similar, and detailed in [22]. So, the neighborhood $O_{\bar{z}}$ is stratified as wanted. The flow is trivially smooth on S_0, it has also been proven that it is smooth on S^s. The only remaining thing to prove is the smoothness on S_0^s. Define the contact time with Σ_0, $\bar{t}(z_0) = \int_0^\infty \rho(s_2, z_0) ds_2$ for $z_0 \in S_0^s$. Then, by dominated convergence, the map $z_0 \in S_0^s \mapsto \bar{t}(z_0)$ is smooth, and so is $z_0 \in S_0^s \mapsto z(\bar{t}(z_0), z_0)$, the contact point with Σ_0. In the end, until $\bar{t}(z_0)$ the flow is smooth, because no singularity occurred yet. For times $t > \bar{t}(z_0)$, one can note that $z(t, z_0) = z(t - \bar{t}(z_0), z(\bar{t}(z_0), z_0))$, which corresponds to the singular flow in Σ_0, and this flow is smooth by proposition 2.2: we have smoothness on S_0^s. The continuity is obtained by the same proof than in the Σ_- case, see [22].

Proof of proposition 2.4

Depending on the sign of H_{12}, the control does not have the same regularity. In the coordinates of system (2.2.1), when $t < \bar{t}$, $u(t) = (\cos\theta(t), \sin\theta(t))$, but from proposition 2.2, when $t > \bar{t}$, $u(t) = u_s(t) = \frac{(-H_{02}, H_{01})}{H_{12}} = \frac{r(-\sin\phi, \cos\phi)}{H_{12}} = \frac{r}{H_{12}}(\cos(\phi + \pi/2), \sin(\phi + \pi/2))$. In the first alternative, the extremal reaches the singular locus at the equilibrium point in the time ρdt, and we have $\theta(\bar{t}) - \phi(\bar{t}) = \pi/2$: the control is continuous when the connexion with the singular flow occurs. In the second one, $\theta(\bar{t}) - \phi(\bar{t}) = -\pi/2$, so that $\theta(\bar{t}^-) = \theta(\bar{t}^+) + \pi$. □

Remark 2.3
From the phase portrait of figure 2.4, we can actually retrieve all three cases. Indeed, one can make a change of coordinates to integrate the parameters α. More precisely, set $\tilde{\xi}_1 = a(\xi) - 1$. Furthermore, the cases Σ_- can be seen as the West part of the phase portrait above the sphere S_+^2, the two lines of zeros corresponding to the partially hyperbolic equilibrium of [22], to retrieve the global phase portrait one has to quotient the s axis to keep s in $\mathbb{S}^1$. The Est part above S_+^2 being the Σ_+ case. The dynamics is actually structurally stable, and the whole situation is contained in the nilpotent case Σ_0.

The following example is close to the nilpotent approximation of the minimum time Kepler problem proposed in [18, 22].

Example 2.1
Let us exhibit a control-affine system with the kind of trajectory describe in theorem 2.2 when the final time is minimized.

Consider

$$\begin{cases} \dot{x}(t) = F_0(x(t)) + u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)), \quad t \in [0, t_f], \quad u \in B(0,1) \\ x(0) = x_0 \\ x(t_f) = x_f \\ t_f \to \min . \end{cases} \tag{2.2.11}$$

on $\mathbb{R}^4$ *with*

$$\begin{cases} F_0(x) = x_1 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4}, \\ F_1(x) = x_2 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \\ F_2(x) = \frac{\partial}{\partial x_2}. \end{cases}$$

Then

$$\textit{rank}(F_1(x), F_2(x), F_{01}(x), F_{02}(x)) = 4, \ \forall x \in \mathbb{R}^4 \backslash \{x_2 = 0\}.$$

The maximized Hamiltonian is $H^{\max}(x,p) = p_3 x_1 + p_4 x_2 + \sqrt{(p_1 x_2 + p_3)^2 + p_2^2}$ *and we have*

$$\begin{cases} \dot{x}_1 = \frac{(p_1 x_2 + p_3) x_2}{\sqrt{((p_1 x_2 + p_3)^2 + p_2^2)}}, & \dot{x}_3 = x_1, \\ \dot{x}_2 = \frac{p_2}{\sqrt{((p_1 x_2 + p_3)^2 + p_2^2)}}, & \dot{x}_4 = x_2. \end{cases} \tag{2.2.12}$$

The coordinates x_3 *and* x_4 *are cyclic, so* p_3 *and* p_4 *are constant. Denote* $p_3 = -a$, $p_4 = -c$, *we get* $p_1(t) = at + b$, $p_2(t) = ct + d$, *where* $b = p_1(0)$, $d = p_4(0)$. *Eventually:*

$$\dot{x}_2 = \frac{ct + d}{\sqrt{((at + b)x_2 - a)^2 + (ct + d)^2}}. \tag{2.2.13}$$

We also have $\Sigma = \{p_2 = p_1 x_2 + p_3 = 0\}$, *and the condition* $H_{01}^2 + H_{02}^2 = H_{12}^2$ *gives* $\Sigma_0 = \Sigma \cap \{p_1^2 = p_3^2 x_2^2 + p_4^2\}$. *The contact time with* Σ *has to be* $\bar{t} = -\frac{d}{c}$. *At* $\bar{t}$, *we must have* $x_2(\bar{t}) = -p_3/p_1(\bar{t}) = -\frac{ac}{ad - bc}$. *In order to reach* Σ_0, *we shall have* $x_2^2(\bar{t}) = \frac{1}{a^2}(p_1^2(\bar{t}) - p_4^2) = \frac{(ad-bc)^2 - c^2}{a^2 c^2} := \bar{x}$. *This gives an equation on the initial conditions* (a, b, c, d)*:*

$$(ad - bc)^2[(ad - bc)^2 - c^2] - a^4 c^4 = 0, \tag{2.2.14}$$

choosing a *and* c *non-zero. This condition imposes* $z(\bar{t}, z_0) \in \Sigma \Rightarrow z(\bar{t}, z_0) \in \Sigma_0$. *Now note that* x_2 *verifies a real ordinary differential equation (though, time dependent)* $\dot{x}_2 = f(t, x_2)$ *with* f *defined by (2.2.13).* f *is regular on* $\mathbb{R}^2 \backslash \{(\bar{t}, \bar{x})\}$. *To* *regularize the dynamics of* x_2 *set* $\mathrm{d}t = \sqrt{((at+b)x_2 - a)^2 + (ct+d)^2}\mathrm{d}s$ *to obtain a continuous dynamical system in the plane:*

$$\begin{cases} x_2' = ct + d \\ t' = \sqrt{((at+b)x_2 - a)^2 + (ct+d)^2} \end{cases}$$

$(\bar{x}, \bar{t})$ *is its only equilibrium. Outside of it,* $t' > 0$*. Choosing* $c > 0$*, there exists a one dimensional stable manifold going to* $(\bar{x}, \bar{t})$*, and thus a trajectory converging to it in infinite time* s*. This implies the existence of a trajectory for (2.2.13) such that* $x_2(\bar{t}) = \bar{x}$*. Hence, together with condition (2.2.14), there exists an extremal reaching* Σ_0*.*

2.3 Conclusion

We treated only the case - although generic - of systems satisfying the assumption (A). Even though we succeeded to give precise result on their behavior, some questions remain open. Those results concern only the necessary conditions for optimality, and at this point, nothing ensures us that extremal trajectories are indeed optimal. Of course, with existence and uniqueness results, if an optimum exists, it is the extremal trajectory studied above. The question of global existence of optimal trajectory can be handled through Filipov's theorem recalled in the next chapter. Unfortunately, without strong assumption, the compactness condition on the state has never been proved for orbit transfer or rendez-vous problem in space mechanics for instance. Another approach to this sufficient conditions for optimality is local: Is the extremal trajectory optimal with respect to all the close admissible $\mathcal{C}^0$ curves. This is the topic of the next chapter.

Chapter 3

Sufficient conditions for optimality of minimum time control-affine systems

Abstract

In this chapter, we prove a sufficient condition for optimality in the case of minimum time affine control systems with double-input control on a 4 dimensional manifold. The proof is based on symplectic methods, and the condition given can be check via a simple numerical test. No strong Legendre-type condition is required.

In this chapter we deal with the notion of sufficient conditions for optimality of our extremals. This topic is still a very active field of research, and a variety of different approaches exist and have been applied to a large number of problems. Geometric methods hold a fairly important place in that regard. Since we have existence and uniqueness of the solutions of the extremal system, to obtain a global result about their optimality, one would like to apply Filippov's theorem, see [24]. This can not be generally achieved, the compactness condition being too delicate to prove, and we switch to local point of view. When the extremal flow (and thus the maximized Hamiltonian) is smooth, the theory of conjugate points can be applied, and local optimality holds before the first conjugate time, we recall this result below. The points where the extremal ceases to be globally optimal are cut points, in general it is an extremely delicate task to compute cut points and cut loci, though it can be done numerically, as for the averaged minimum energy orbit transfer problem, in the two body case, see [16]. Unfortunately, we rarely encounter the smooth case in practice, and there is a lack of general method overcoming the different kind of singularities. An extension of the smooth case method which uses the Poincaré-Cartan integral invariant, see [4], is easier to generalize to non-smooth cases, and has been used to prove local optimality for L^1 minimization of mechanical systems for instance, in [19]. We use a similar technique to prove theorem 3.2, one of the main difference being the type of singularity: L^1-minimization of the control creates singularities of codimension one, and the extremal flow is the concatenation of the flow of two regular Hamiltonians. In our case, we have codimension two (meaning, unstable) singularities, and one non-Lipschitz Hamiltonian. When the control lies in a box, second order conditions can be of use through a finite dimensional subsystem given by allowing the switching times to variate, those techniques have been initiated by Stefani and Poggiolini, see [1], for instance. The majority of this work proved local optimality for normal extremal, and a few of them tackle the abnormal case. One can cite for instance [56] where single input systems are handled. One can refer as well to [39] where theoretical as well as numerical studies are leaded when the control lies in a polyhedron. We will also tackle only the normal case in the following, since the co-dimension two singularity inducted by minimizing the final time is our main focus in this thesis. The recent paper [3] from Agrachev and Biolo, proved local optimality of these broken extremal around the singularity with extra hypothesis on the adjoint state. Our result present the interest of being more global (in the sens that it is viable along a whole trajectory), and easily checked by a simple numerical test. Thanks to that optimality analysis, we can investigate the regularity of a upper bound to the value function of this time optimal problem: the final time of extremals and prove that it is piecewise smooth.

3.1 The smooth theory

Let us begin by recalling the classical options when the extremal system is smooth. Consider an optimal control system

$$\dot{x} = f(x, u),\ u \in U \tag{3.1.1}$$

and assume $H^{\max}(x, p) = \max\limits_{u \in U} H(x, p, u)$ is smooth. We present a method described in [4], and refined in [19] to deal with codimension a one singularity set. Denote $\bar{z}(t) = (\bar{x}(t), \bar{p}(t))$, $t \in [0, \bar{t}_f]$ the reference extremal, starting from $\bar{z}_0 \in T^*M$, $\bar{u}$ its associated control,and consider the variational equation along $\bar{z}(t)$:

$$\dot{\delta z} = J\nabla^2 H(z(t))\delta z \tag{3.1.2}$$

Solutions of (3.1.2) are called Jacobi fields.

Definition 3.1 (Conjugate times & points)
A time t_c is called a conjugate time if there exists Jacobi field δz such that

$$d\pi(z(0))\delta z(0) = d\pi(z(t_c))\delta z(t_c) = 0$$

(ie, $\delta x(0) = \delta x(t_c) = 0$). We say δz is vertical at 0 and t_c. The point $x(t_c) = \pi(z(t_c))$ is a conjugate point.

The following result imply optimality until the first conjugate time.

Theorem 3.1
Assume:

(1) The reference extremal is normal.

(2) $\frac{\partial x}{\partial p_0}(t, \bar{z}_0) \neq 0$ for all $t \in]0, t_f]$.

then the reference trajectory x is a local minimizer among all the $\mathcal{C}^0$-admissible trajectories with same end points.

Assumption (2) provide disconjugacy along the reference extremal, and can be verified through a simple numerical test. The proof consists in building a Lagrangian manifold, and propagating in by the extremal flow, then one can prove the projection is invertible on this manifold: this allows one to lift all the admissible trajectories with same end points to the cotangent bundle, and one can compare, using the Poincaré-Cartan invariant, their cost with the one of the reference extremal. We will extend this proof to the non-smooth case of our minimum time affine control systems.

3.2 Our setting

Consider the following optimal time control system

$$\begin{cases} \dot{x}(t) = F_0(x(t)) + u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)), \quad t \in [0, t_f], \quad u \in U \\ x(0) = x_0 \\ x(t_f) = x_f \\ t_f \to \min . \end{cases} \tag{3.2.1}$$

with the control set U being the euclidean unitary ball. We still denote $\mathcal{U} = L^\infty([0, t_f], U)$ the set of admissible controls, and make the assumption given in chapter I:

$$\det(F_1(x), F_2(x), F_{01}(x), F_{02}(x)) \neq 0, \text{ for almost all } x \in M. \tag{A}$$

Recall the (non-smooth) maximized Hamiltonian is

$$H^{\max}(z) = H_0(z) + \sqrt{H_1^2(z) + H_2^2(z)},$$

and the singular locus $\Sigma := \{H_1 = H_2 = 0\}$. One can make a comparison between those singularities and the double switchings obtained by taking $U = [-1, 1]^2$ (or even $[-1, 1]^m$). In that regard, it has been proved in [48] that extremals are optimal provided some strong Legendre-type conditions and the coerciveness of a second variation to a finite dimensional problem obtained by perturbation of the switching times. This result holds also for the abnormal case. Our theorem do not require any coerciveness assumption. Σ has a partition into three subsets. No extremal would reach Σ_+, so all extremals around this set are smooth, theorem 3.1 applies. The singular extremals lying inside cannot be optimal via the Goh condition. The minimization of the final time implies $H \equiv -p^0$, where p^0 is the negative constant from theorem 1.2. Thus, for normal extremals, we can set $H \equiv 1$, and along abnormal extremals we have $H \equiv 0$. We will deal with the Σ_- case. According to theorem 2.1, in chapter II, we know the extremal flow z is $P\mathcal{C}^\infty$ in a neighborhood of Σ_-. More precisely, there exist two codimension one submanifolds S^s and S^u (see chapter I) in a neighborhood $O_{\bar{z}}$ of a point $\bar{z} \in \Sigma_-$ such that the map

$$\begin{array}{rcl} z_i : [0, t_f] \times S^i \backslash \Delta & \to & T^*M \\ (t, z_0) & \mapsto & z(t, z_0) \end{array}$$

is $\mathcal{C}^\infty$ smooth ($\Delta = \{(\bar{t}(z_0), z_0),\ z_0 \in S^i\}$, $\bar{t}(z_0)$, $i = s, u$. being the switching time of the extremal from z_0). S^s is the set of initial conditions brought to the singular locus by the flow, S^u is the set a initial conditions converging to Σ in negative times, in other worlds, the image of S^s by the flow for times greater than $\bar{t}(z_0)$.

Proposition 3.1
The limits $\dot{z}(\bar{t}(z_0)_\pm, z_0)$ as well as $\frac{\partial z}{\partial z_0}(\bar{t}(z_0), z_0)$ are well defined.

Proof. $\dot{z}(\bar{t}(z_0)_\pm, z_0)$ are easily defined since the control along an extremal has well defined right and left limits around a switching time. Writing the extremal flow as the integral of the regularized vector field (see chapter I) X: $z(\bar{t}(z_0), z_0) = \int_0^{+\infty} X(\tilde{z}(s, z_0))ds + z_0$ for $z_0 \in S^s$, then the dominated convergence theorem gives $\frac{\partial z}{\partial z_0}(\bar{t}(z_0), z_0) = \int_0^{+\infty} \frac{\partial}{\partial z_0} X(\tilde{z}(s, z_0))ds + I_{2n}$, where $\tilde{z}$ is the flow of X. □

For extremal outside of S^s, the flow of the maximized Hamiltonian is smooth, and the usual sufficient conditions for optimality apply. Let us denote $\bar{z}(t)$ our reference extremal, lying in S^s, with final time $\bar{t}_f$ and $\bar{t} := \bar{t}(z_0)$, $\bar{z}(\bar{t}) = \bar{z}$.

Assumption The fiber $T^*_{x_0}M$ and S^s intersect transversally: $T^*_{x_0}M \pitchfork S^s$, then $T^*_{x_0}M \cap S^s$ is a smooth submanifold of dimension 3. Let us define the exponential mapping

Definition 3.2 (exponential map)
We call exponential mapping from x_0, the application

$$\exp_{x_0} : (t, p_0) \in [0, t_f] \times T^*_{\bar{x}_0}M \cap S^s \to \pi(z(t, x_0, p_0)) = x(t, x_0, p_0) \in M$$

The exponential map is smooth except on Δ, ie, when $x(t, x_0, p_0) \notin \Sigma$. Denote $\bar{t} := \bar{t}(\bar{z}_0)$. The differential of the exponential mapping $d\exp_{x_0}(t, p_0) = (\dot{x}, \frac{\partial x}{\partial p_0})(t, p_0)$ is a 4×4 matrix, where $\frac{\partial}{\partial p_0}$ denote the derivation with respect to a set of coordinates on $T^*_{x_0}M \cap S^s$, and $M(t) := d\exp_{\bar{x}_0}(t, \bar{p}_0)$. Under our transversality assumption, we have

Theorem 3.2
Assume

(A_0) *The reference extremal is normal,*

(A_1) $\det M(t) \neq 0$ *for all* $t \in]0, \bar{t}[\cup]\bar{t}, \bar{t}_f]$ *and* $\det M(\bar{t}_-) \det M(\bar{t}_+) \neq 0$,

then the reference trajectory is a $\mathcal{C}^0$*-local minimizer among all trajectories with same endpoints.*

Obviously when $t = 0$, $\frac{\partial x}{\partial p_0}(0, \bar{z}_0) = 0$, and some part of the proof is dedicated to extend condition (A_1) to $t = 0$.

3.3 Proof of theorem 3.2

The rest of the chapter is devoted to prove theorem 3.2.

Lemma 3.1
Condition (A_1) *implies that there exists a Lagrangian submanifold* $\mathcal{L}$ *transverse to* $T^*_{x_0}M$*, and close enough to* $T^*_{x_0}M$ *so* $\mathscr{S}_0 = \mathcal{L} \cap S^s$ *is a smooth submanifold of dimension 3, and such that* $(\frac{\partial x}{\partial t}, \frac{\partial x}{\partial z_0})(t, \bar{z}_0)$ *is invertible on* $[0, \bar{t}[\times \mathscr{S}_0$*, as well as on* $]\bar{t}, t_f] \times \mathscr{S}_0$*; where* z_0 *denote coordinates on* $\mathscr{S}_0$*.*

Also, $(\frac{\partial x}{\partial t}, \frac{\partial x}{\partial z_0})(\bar{t}_-, \bar{z}_0)$ and $(\frac{\partial x}{\partial t}, \frac{\partial x}{\partial z_0})(\bar{t}_+, \bar{z}_0)$ are invertible. Thus, the canonical projection π is a diffeomorphism from $z(]0, \bar{t}[, \mathscr{S}_0)$ onto its image and an homeomorphism from

$$\mathscr{S}_1 = \{z(t, z_0),\ (t, z_0) \in [0, \bar{t}(z_0)] \times \mathscr{S}_0\} \tag{3.3.1}$$

onto its image. The same holds for

$$\mathscr{S}_2 = \{z(t, z_0),\ (t, z_0) \in [\bar{t}(z_0), t_f] \times \mathscr{S}_0\}. \tag{3.3.2}$$

Let us prove that π is a homeomorphism on their union. To show that we can glue those together, it is sufficient to prove that the extremal passes transversally through $\Sigma_1 := \pi(\Sigma) \cap \mathscr{S}_1$. Since the map $(t, z_0) \in \mathbb{R} \times \mathscr{S}_0 \mapsto x(t, z_0) \in \pi(\mathscr{S}_1)$ is a homeomorphism, and is differentiable for all $(t, z_0) \neq (\bar{t}(z_0), z_0)$ with well defined limits, we can define its inverse function $z_0(t, x)$, and $f(t, x) = t - \bar{t}(z_0(t, x))$. Thus we have $\Sigma_1 = \{f = 0\}$. Now denote $g(t) = f(t, \bar{x}(t))$, we get

$$\dot{g}(\bar{t}_-) = 1 = \dot{g}(\bar{t}_+) - d\bar{t}(\bar{z}_0)\left[\frac{\partial z_0}{\partial t} + \frac{\partial z_0}{\partial x}\dot{\bar{x}}(t)\right], \tag{3.3.3}$$

but $\frac{\partial z_0}{\partial t}(t, z_0(t,x))) = -\left(\frac{\partial x}{\partial z_0}\right)^{-1} \dot{x}(t, z_0(t,x))$. Hence, we obtain $\dot{g}(\bar{t}_-) = \dot{g}(\bar{t}_+) = 1$. Meaning that, in a neighborhood of $\bar{z}$, every extremal passes transversally through $\pi(\Sigma_1)$: by restricting $\mathscr{S}_0$ if necessary, every extremal from $\mathscr{S}_0$ passes transversally through $\pi(\Sigma_1)$, and the projection defines a continuous bijection on $\mathscr{S}_1 \cup \mathscr{S}_2$, and even a homeomorphism if we restrict ourselves to a compact neighborhood of the reference extremal. Denote π^{-1} its inverse function.

We will now prove that the Liouville form $\lambda = pdx$ is exact on $\mathscr{S}_1$ and $\mathscr{S}_2$. Let us first prove that λ is closed on $\mathscr{S}_i$. Tangent vectors to $\mathscr{S}_1$ at (t,z) are parametrized as followed:

$$(\delta t, \dot{z}(t, z_0)\delta t + \frac{\partial z}{\partial z_0}(t, z_0)\delta z_0), \text{ with } (\delta t, \delta z_0) \in \mathbb{R} \times T_{z_0}\mathscr{S}_0,\ z(t, z_0) = z,$$

whenever $(t,z) \neq (\bar{t}(z_0), z(\overline{t(z_0)}, z_0)$. In that last case, tangent vectors are given by

$$(\delta t, \dot{z}(t_-, z_0)\delta t + \frac{\partial z}{\partial z_0}(t_-, z_0)\delta z_0), \text{ with } (\delta t, \delta z_0) \in \mathbb{R} \times T_{z_0}\mathscr{S}_0,\ z(t, z_0) = z.$$

Let $(v_1, v_2) \in T\mathscr{S}_i$, we have

$$d\lambda(v_1 v_2) = dp \wedge dx(\frac{\partial z}{\partial z_0}(t, z_0)\delta z_0^1, \frac{\partial z}{\partial z_0}(t, z_0)\delta z_0^2) = \omega(\delta z_0^1, \delta z_0^2), \tag{3.3.4}$$

because the flow is symplectic on S^s, and, since $\mathscr{S}_0 \subset \mathcal{L}$ which is Lagrangian, $\omega(\delta z_0^1, \delta z_0^2) = 0$. This equality still holds for tangent vectors at $(\bar{t}(z_0), z(\overline{t(z_0)}, z_0)$. Being closed, the Liouville form is actually exact on each $\mathscr{S}_i$. Indeed, consider a curve $\gamma(s) = (t(s), z(t(s), z_0(s))$ on $\mathscr{S}_1 \cup \mathscr{S}_2$, it actually retracts continuously on $\gamma_0(s) = (0, z_0(s))$. Then, since λ is closed,

$$\int_\gamma \lambda = \int_{\gamma_0} \lambda,$$

though, locally, one can chose $\mathcal{L}$ as the graph of the differential of a smooth function, so we end up with

$$\int_{\gamma_0} \lambda = 0,$$

by Stokes formula. Now, let us prove that our reference extremal $\bar{z} = (\bar{x}, \bar{p})$ minimizes the final time among all close $\mathcal{C}^1$-curves with same endpoints. Let $x(t)$, $t \in [0, t_f]$ be a admissible curve of $\mathcal{C}^1$ regularity, generated by a control u with

$x(0) = x_0$, $\mathcal{C}^0$ close to $\bar{x}$, then, denote $z(t) = (x(t), p(t))$ its well defined lift in $\mathscr{S}_1 \cup \mathscr{S}_2$ by π^{-1}. Since λ is exact, we have

$$\int_z pdx = \int_{\bar{z}} pdx$$

Thus,

$$\int_0^{t_f} p(t)\dot{x}(t)dt = \int_0^{\bar{t}_f} \bar{p}(t)\dot{\bar{x}}(t)dt.$$

The left hand side gives

$$\begin{aligned}\int_0^{t_f} \langle p(t), \dot{x}(t)\rangle dt &= \int_0^{t_f} \langle p(t), F_0(x(t)) + u_1(t)F_1(x(t)) + u_2F_2(x(t))\rangle dt \\ &= \int_0^{t_f} H(x(t), p(t), u(t))dt \leqslant \int_0^{t_f} H^{\max}(x(t), p(t))dt = t_f\end{aligned}$$

since $H^{\max}(x(t), p(t)) \equiv H^{\max}(x(0), p(0)) = H^{\max}(\bar{x}_0, \bar{p}_0) = 1$, but the right hand side is actually just

$$\int_0^{\bar{t}_f} H^{\max}(\bar{x}, \bar{p})dt = \bar{t}_f,$$

so that $t_f \geqslant \bar{t}_f$. We have the desired optimality among all $\mathcal{C}^0$-close curves with $\mathcal{C}^1$ regularity. To extend the result to all close admissible curves, consider such a curve $\tilde{x}$ with same endpoints and denote $\tilde{u}$ its associated control. Take a $\mathcal{C}^1$ curve x in a small enough neighborhood of $\tilde{x}$ in the $W^{1,\infty}$ topology (admissible curves are absolutely continuous). $\tilde{x}$ and x actually have the same cost t_f. The previous analysis applying to x, the result holds for all $\mathcal{C}^0$-close admissible curves. □

If $T^*_{x_0}M$ and S^s do not intersect each other (which is also transversality), then the extremal flow is smooth in a neighborhood of the reference extremal, and the smooth theory can be applied. Thus, an answer is provided generically. In the very specific case when $T^*_{x_0}M \subset S^s$, one has to change a bit the exponential mapping defined above, but the same proof basically holds. The only remaining case, when the cotangent fiber at x_0 and S^s intersect non-transversally.

Proof of Lemma 3.1. We follow and adapt the proof in annex of [19]. Let S_0 be a symmetric matrix so that the Lagrangian subspace $L_0 = \{\delta x_0 = S_0 \delta p_0\}$ intersects transversally with $T_{\bar{z}_0} S^s$. Consider the two linear symplectic systems

$$\delta\dot{z}(t) = \frac{\partial H^{\max}}{\partial z}(\bar{z}(t))\delta z(t),\ t \in [0, \bar{t}[, \delta z(0) = (S_0, I)$$

and

$$\dot{\phi}(t) = \frac{\partial H}{\partial z}(\bar{z}(t), u(t))\phi(t),\ t \in [0, \bar{t}[, \phi(0) = (I).$$

Then set $\delta\tilde{z}(t) = (\delta\tilde{x}(t), \delta\tilde{p}(t)) = \phi(t)^{-1}\delta z(t)$. Since $\delta\tilde{z}(0) = \delta z(0) = (S_0, I)$, the matrix

$$S(t) = \delta\tilde{x}(t)\delta\tilde{p}(t)^{-1}$$

exists for small enough t. It is symmetric since

$$L_t = \exp(X_{H^{\max}}t)'(L_0) \text{ and } (\phi(t))^{-1}(L_t)$$

are Lagrangian submanifolds. One can prove that $\dot{S}(t) \geqslant 0$ (see [19], annex), whenever $S(t)$ is defined, as the consequence of the classical first and second order conditions on the maximized Hamiltonian. Then, if $S_0 > 0$ (small enough so that $S(t)$ is defined on $[0, \varepsilon]$, $S(t)$ is invertible, and as such, $\phi(t)^{-1}(L_t) \pitchfork \ker d\pi(\bar{z}_0)$. This implies $L_t \pitchfork \ker d\pi(\bar{z}(t))$ since $\phi(t)(\ker d\pi(\bar{z}_0)) = \ker d\pi(\bar{z}(t))$. There exists a Lagrangian submanifold $\mathcal{L}_0$ of T^*M tangent at L_0 in $\bar{z}_0$. It intersects S^s transversally, and the lemma follows. □

3.4 Regularity of the field of extremal

Fix $x_0 \in M$, the value function associate to a final state the optimal cost, and is define as followed:

$$S_{x_0} : x_f \in M \mapsto \inf_{u \in \mathcal{U}} \{t_f,\ x(t_f, u) = x_f\} \in \mathbb{R}.$$

It defines a pseudo-distance between x_0 and x_f and its regularity is a crucial information in optimal control problem (especially in sub-Riemannian geometry, where is simply defines the distance). We give the regularity of the final time for extremal, which, the conditions of the previous part are locally optimal. If they are globally optimal (which is true for small enough times), it coincide with the value function, otherwise, we only obtain the regularity of an upper bound to the value function. Actually, since the differential equation is homogeneous in the adjoint vector, one can restrict to the unitary bundle of the cotangent bundle ST^*M, and consider

$$\exp^1 : (t_f, p_0) \in \mathbb{R}_+ \times ST^*_{x_0}(M) \mapsto x(t_f, x_0, p_0) = x_f \in M.$$

We proved in chapter 1 that this function is piecewise smooth, and belongs to the log-exp category (theorem 1.5). Now we have two cases:

First case In the neighborhood of $(x_0, \bar{p}_0) \in S^s$, the extremal flow, as well as F, are smooth. If $dF(\bar{t}_f, \bar{p}_0)$ is invertible, ie, if $\det(\dot{x}(\bar{t}_f, x_0, \bar{p}_0), \frac{\partial x}{\partial p_0}(\bar{t}_f, x_0, \bar{p}_0)) \neq 0$, for all t, where p_0 is a system of coordinates on $ST^*(M)$ around $(x_0, \bar{p}_0)$, then, locally, we have a C^1 inverse $F^{-1}(x_f) = (t_f, p_0)(x_f)$. This is the well-known smooth case.

Second case In the neighborhood $(x_0, \bar{p}_0) \in S^s$, then instead of $\exp^1$, chose the exponential mapping of definition 3.2:

$$\exp : (t_f, p_0) \in \mathbb{R}_+ \times (S^s \cap T^*_{x_0}M) \mapsto x(t_f, x_0, p_0) = x_f \in M.$$

Under the transversality condition, $S^s \cap T^*_{x_0}M$ is a nice 3 dimensional submanifold, and since the flow is smooth on S^s, the same process can be applied with the same result, except when $x_f \in \Sigma$. In such a neighborhood, we only have P$\mathcal{C}^1$ regularity for G, and we need a weaker inverse function theorem. A certain amount of such results exist in the literature, we will use a theorem from [31].

Theorem 3.3
If assumptions

(A_1) $\det(\dot{x}(\bar{t}_f, x_0, \bar{p}_0), \frac{\partial x}{\partial p_0}(\bar{t}_f, x_0, \bar{p}_0)) \neq 0$ *for all* $t \neq \bar{t}$ *and*

(A'_2) $\det(\dot{x}(\bar{t}_{f-}, x_0, \bar{p}_0), \frac{\partial x}{\partial p_0}(\bar{t}_{f-}, x_0, \bar{p}_0)).\det(\dot{x}(\bar{t}_{f+}, x_0, \bar{p}_0), \frac{\partial x}{\partial p_0}(\bar{t}_{f+}, x_0, \bar{p}_0)) > 0,$

hold, then the final time $x_f \mapsto t_f(x_f)$*, is continuous and piecewise* $\mathcal{C}^1$ *in a neighborhood of* $x(\bar{t}_f, x_0, \bar{p}_0)$*.*

Proof. Thanks to (A_1) and (A'_2) we have a PC^1 inverse, by theorem 3 in [31] for instance, $x_f \mapsto (t_f(x_f), p_0(x_f))$ is piecewise $\mathcal{C}^1$. □

Obviously (A'_2) implies (A_2) and the extremal is locally optimal by theorem 3.2 above. In the case it is globally optimal, the value function is just $S(x_f) = t_f(x_f)$, the final time of the extremal. Otherwise, $S(x_f) \leqslant t_f$ and we only have PC^1 regularity for an upper bound function to the value function: the final time of extremal trajectories.

3.5 Conclusion and perspectives

Sufficient conditions for optimality rely strongly on methods from symplectic geometry. We proved a local optimality condition for minimum time extremals of control-affine systems, with double input controls on a 4 dimensional manifold:

this result, as theorem 2.1 can actually be extended to a $2n$ dimensional manifold with n controls with a natural modification of assumption (A) given in the conclusion of chapter I.

One of the main issue is, more than often in practice, the irregularities of the maximized Hamiltonian. This, implying irregularities for the extremal flow itself, is making the goal of a general theory of sufficient conditions for optimal control problems a very delicate task. Some hope in that matter are provided by symplectic topology and singular symplectic geometry: the theory of Hamiltonian homeomorphisms and b^m-symplectic structures. Indeed, some flow from optimal control problem are not smooth, but are the uniform limit of smooth Hamiltonian flows, this is the definition of an Hamiltonian homeomorphism. The image by such a map of a Lagrangian submanifold is still a Lagrangian submanifold, which is a very interesting property when proving sufficient conditions for optimality. Finally, through non-smooth changes of coordinates, one can regularize the vector field, and thus the extremal flow, but destroys the symplectic structures. The image of the symplectic form is not a differential form anymore, but turns out, in some cases, to have very similar properties. Those cases are describe by b-(or b^m or sometimes, log) symplectic structures, see [17] for instance. This is the type of structures obtained when one perform McGehee coordinate change for the three body problem for instance. This geometric setting could allow the author to build a general theory by making a classification of the singularities met in the maximized Hamiltonians.

Chapter 4

Liouville integrability and optimal control systems

Abstract

In this chapter, we give a proof of the non-integrability of the minimum time Kepler problem. The proof rests on Galois differential theory methods and Moralès-Ramis theorem. After introducing the tools, we find a particular solution - a collision orbit, and study the variational equations along this orbit. We prove that the Galois group of this equation contains $SL_2(\mathbb{C})$.

4.1 Liouville-integrability and algebraic obstructions

4.1.1 Integrability of Hamiltonian systems and optimal control theory

Preliminaries

In this section (M, ω) will denote a smooth symplectic manifold of dimension $2n$, with symplectic form ω. A diffeomorphism ϕ is said to be symplectic if it preserve the symplectic form: if $\phi_*\omega = \omega$. We recall now the necessary prerequisite about Liouville integrability of Hamiltonian systems, see [5] Classical, a smooth function $H \in \mathcal{C}^\infty(M)$ will be called a Hamiltonian. Denote $\Gamma(TM)$ the space of sections of the tangent fiber. For any $X \in \Gamma(TM)$, we denote i_X the inner product.

Definition 4.1
Any Hamiltonian H defines (uniquely) a vector field X_H on M by

$$dH = i_{X_H}\omega.$$

Uniqueness is provided by the non-degeneracy of ω. The Hamiltonian system associated to H is the differential equation

$$\dot{z} = X_H(z). \tag{4.1.1}$$

We call integral curves, the solutions of (4.1.1). In local Darboux coordinates (p, q) on M, $\omega = dp \wedge dq$, and the classical Hamilton's equation of motion are:

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q}. \end{cases} \tag{4.1.2}$$

Let $J_n = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$, then, denoting $z = (q, p)$, (4.1.2) $\Leftrightarrow \dot{z} = J_n \nabla H(z)$. M being of dimension $2n$, we say that H has n *degrees of freedom*. The cotangent bundle of a smooth connected manifold X may be the most natural case of symplectic structure: $M = T^*X$ has a canonical exact symplectic form, $\omega = d\lambda$ where $\lambda = pdq$ is know as the Liouville form. By the necessary conditions given by the PMP, this is the framework in which we always work in geometric optimal control.

Definition 4.2
A non constant function $f \in \mathcal{C}^\infty(M)$ is said to be a first integral of (4.1.1) if and only if f is constant along all its integral curves.

When M is symplectic, $\mathcal{C}^\infty(M)$ is actually a Poisson algebra, with Poisson bracket $\{f,g\} = \omega(X_f, X_g)$, $\forall f, g \in \mathcal{C}^\infty(M)$. In Darboux coordinates, we have the formula

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}.$$

If $\{f,g\} \equiv 0$, we say f and g are in *involution*. Since $\frac{\mathrm{d}}{\mathrm{d}t} f(\phi^t) = \{f,H\}(\phi^t)$, we have

Proposition 4.1
f is a first integral of (4.1.1) if and only if $\{H,f\} = 0$.

Obviously, H itself is a first integral of its own Hamiltonian system. Denoting, ${\phi_H}^t$ (or just ϕ if there is no ambiguities) the flow of (4.1.1), we have $({\phi_H}^t)_*\omega = \omega$ for all t: an Hamiltonian flow is symplectic. ω^n is a volume form on M, so a symplectic diffeomorphism is also volume preserving. The default of a symplectic map to be Hamiltonian is a very important notion in modern symplectic geometry.

Any mechanical system: $\ddot{q} = -\nabla U(q)$, where $U : \mathbb{R}^n \to \mathbb{R}$ is smooth, is Hamiltonian with $H(p,q) = \frac{p^2}{2} + U(q)$.

Example 4.1
When $n = 1$, $U(q) = \cos q$ defines the motion of a pendulum: $\ddot{q} - \sin q = 0$. Having only one degree of freedom, besides H this system cannot have any additional independent first integral.

Example 4.2
If $n = 2$, $U(q) = -\frac{1}{\|q\|}$ ($\|.\|$ being the euclidean norm), H defines the famous Kepler problem (we saw the controlled version in example 1.1), describing the reduction of the motion of two bodies attracting each others by the Newtonian gravitational force in the plane. If $H < 0$, they are in elliptic motion around their center of mass, and each body has the dynamics given by H. The angular momentum (or rather its only non zero component) $C = q_1p_2 - q_2p_1$ is a first integral of this system.

Example 4.3
The free (as opposed to controlled, see example 1.2) restricted elliptic (resp. circular) three body problem, modeling the motion of a massless body under the influence of two others, called the primaries, which move along elliptic (resp. circular) planar Keplerian orbits. Denote μ, the ratio of the masses of the two primaries, their dynamics is:

$$\ddot{q} + \nabla V_\mu(t,q) = 0,$$

with $V_\mu(t,q) = \frac{1-\mu}{|q-q^1(t)|} + \frac{\mu}{|q-q^2(t)|}$ *and* q^1 *and* q^2 *being the position vectors of the primaries.*
The circular case (RCTBP) admits a useful autonomous formulation

$$\ddot{q} + (1-\mu)\frac{q+\mu}{|q+\mu|^3} + \mu\frac{q-1+\mu}{|q-1+\mu|^3} + 2J\dot{q} - q = 0,$$

in the rotating frame. Those those systems both admit the angular momentum as a first integral. In this thesis we will be interested in their optimal control, a force u*, modeling the thrust of the third body, will be added to the right hand side.*

Under the right conditions (for instance, convexity in p), solutions of Hamilton's equations are minimizers of a variational principle given by the Lagrangian L, which is defined on the tangent bundle by the Legendre transform of the Hamiltonian:

$$L(x,v) = \sup_{p \in T^*M} pv - H(x,p).$$

This definition motivates Pontrjagin's *pseudo-Hamiltonian* in optimal control.

Arnold-Liouville theorem and action-angle coordinates for integrable systems

We now recall some facts of the theory of integrable systems, starting by the definition of integrability.

Definition 4.3
Let H *be a* n *degrees of freedom Hamiltonian. We say (4.1.1) is Liouville integrable if it exist* n *independent - in the sense that their gradient are almost everywhere linearly independent - first integrals* $(f_1 = H, f_2, \dots, f_n)$ *in involution.*

An integrable system has enough first integrals which commutes for the Poisson bracket. This property implies strong structural result as we will see from Arnold-Liouville theorem. The first one, obvious, is that each integral curves are contained in one of the n-dimension manifolds $M_{\mathbf{c}} = \{(f_1,\dots,f_n) = (c_1,\dots,c_n)\}$, $\mathbf{c} = (c_1,\dots,c_n) \in \mathbb{R}^n$. On each of these manifold, the motion of an integrable Hamiltonian system is almost as simple as a periodic motion; it is has actually several periods in the following sens: in coordinates $\phi^t = (\phi^t_i)_i$, for each i there exists T_i such that $\phi_i^{t+T_i} = \phi_i^t$. We say the motion is quasi-periodic.

Theorem 4.1 (Arnold-Liouville, [5])
If H is integrable, then the M_c's are smooth submanifolds stable by the flow, foliating the phase space M. If M_c is compact and connected, then it is diffeomorphic to a n-dimensional torus $\mathbb{T}^n$, and the motion on each torus is quasi-periodic.

Quasi-periodic motions are extremely important in Hamiltonian dynamics since they are extremely simple, express in a suitable set of coordinates, called *action-angle*. Indeed, we have

Theorem 4.2 (Action-angle coordinates)
In a neighborhood of a compact connected leaf M_c, there exists coordinates (I, φ) on M, such that $\omega = dI \wedge d\varphi$ (canonical) and $H = H(I)$, implying

$$\dot{I} = 0, \qquad \dot{\varphi} = -\frac{\partial H}{\partial I} = \omega(I).$$

The action variables I parametrize the torus on which the motion takes place. The angles $\varphi(t) = \varphi(0) + \omega(I)t$ describe the motion on this torus, which actually consists of straight lines in the universal covering. Two important cases are to be distinguished: if the components of $\omega(I)$ are rationally dependent, then trajectories on the torus are periodic. Otherwise, they are dense, and in this last case the torus may be preserved by arbitrary small perturbations, see [5, 30] for instance.

Example 4.1 is obviously integrable since it has one degree of freedom. It can be integrated via Jacobi elliptic functions. The Kepler problem of example 4.2 is integrable as well, having an extra independent first integral C, it has to commute with the Hamiltonian. According to Arnold-Liouville theorem, the trajectories are evolving on a 2 torus. Actually we have more: One can find another independent first integral, for instance, the eccentricity e of the conic, which does not commute with the two others. The Kepler problem is said to be super integrable: The trajectories, when compact, are actually constrained on one dimensional tori, and thus, are periodic (that is, for $0 \leqslant e < 1$ or $H < 0$).

Proving the integrability property by finding new first integrals can be a delicate task, as well as proving the non-integrability of a system. One of the first example of non integrability proof is the celebrated work of Poincaré in [50] for the three body problem. Since, a variety of techniques have been developed, using tools from algebra and topology, by Ziglin with the study of monodromy, and more recently, by Ramis and Moralès and their differential Galois theory.

4.1.2 Introduction to Galois Differential Theory

Galois differential theory was developed as a systematic way to prove or deny integrability of linear differential equation. We recall some of the main tools and definitions, one can see [54, 59], and then [40] for the details and proofs.

The linear theory of differential Galois group

A differential field (k, ∂) is a field endowed with a linear map $\partial : k \to k$ verifying Leibniz rule: $\partial(ab) = b\partial a + a\partial b$. In that whole section, (k, ∂) will be a differential field of characteristic zero, whose field of constants $k_0 = \ker \partial$ is algebraically closed. We will often use the notation ∂x instead of x', $x \in k$. In general, k can be either the field of meromorphic functions $\mathcal{M}(\mathbb{C})$, the field of rational complex functions $\mathbb{C}(z)$ or the field of Laurent series $\mathbb{C}\{\{z^{-1}\}\}[z]$, for instance.

Consider a linear differential equation

$$(L) : Y' = AY, \; A \in M_n(k).$$

We want the Galois group of (L) to be the group of symmetries preserving all algebraic and differential relations of a basis of solutions, by analogy with the classical Galois group of a polynomial equation. The space of solution $V = \{v \in k^n, \; v' = Av\}$ is a vectorial space over k_0 of dimension less than n. A fundamental matrix of (L) is a invertible matrix $Z \in Gl_n(k)$ such that $Z' = AZ$. This matrix only exists if $\dim V = n$, that is why we need the next definition of a larger ring, such that we always have a fundamental matrix, the Picard-Vessiot ring.

Definition 4.4 (Picard-Vessiot ring)
A Picard-Vessiot ring for $Y' = AY$ is a differential ring R over k such that

(i) The only differential ideals of R are (0) and R.

(ii) There exists a fundamental matrix $Z \in GL_n(R)$ for the equation $Y' = AY$.

(iii) R is generated as a ring by k, the entries of Z and $1/det(Z)$.

We can construct the Picard-Vessiot ring by the following process:

Consider the polynomial ring

$$S = k[Y_{1,1}, \ldots, Y_{n,n}, 1/\det(Y)]$$

where Y is an $n \times n$ matrix. This ring has a derivation provided by the differential system $Y' = AY$. Let $\mathcal{I}$ be a maximal differential ideal of S, and the quotient

$R = S/\mathcal{I}$. It turns out that the choice of the maximal differential ideal M always gives the same Picard-Vessiot ring up to isomorphism. This ring is also a domain, thus allowing to consider the quotient field, the Picard-Vessiot field, denoted $K = \mathrm{Frac}(k)$. The same definitions apply to scalar differential equation of any order: $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$, modulo the use of the companion matrix. It is now time to define the fundamental object of this theory

Definition 4.5 (Galois group)
The Galois group $\mathrm{Gal}(R/k)$ *of the linear differential equation* (L)*, or scalar equation given by L, is the group of differential* k*-algebra automorphisms of* R*: automorphisms fixing the elements of* k *that commute with the differentiation* ∂*.*

Given a fundamental matrix of solution Z and a Galois group element σ, we have $Z' = AZ$, and thus applying σ, we also have $\sigma(Z)' = A\sigma(Z)$. Thus $\sigma(Z)$ is also a matrix of solutions; there exists a constant matrix C such that $\sigma(Z) = ZC$, and as σ is an automorphism, C has to be invertible. As a result, there is an injection $\mathrm{Gal}(R/k) \hookrightarrow Gl_n(k_0)$, can be represented as a subgroup of invertible $n \times n$ matrices. Even more, we have

Proposition 4.2
The Galois group $\mathrm{Gal}(R/k) \subset GL_n(k_0)$ *is a linear algebraic group, i.e. the zero set in* $Gl_n(k_0)$ *of a system of polynomials over* k_0 *in* n^2 *variables.*

Proof. This can be obtained by letting a Galois group element σ act (right multiplication by a matrix) on the differential ideal $I = (f_1, \ldots, f_p)$. We can moreover assume that $f_i \in k[Y]$. As this does not change the degrees in the $Y_{i,j}$ and since I must be stabilized, $\sigma(f_i)$ must belong to $I \cap k_{\max(\deg f_1,\ldots,\deg f_p)}[Y]$. This condition is a condition of membership to a vector space, which provides algebraic conditions on the entries of the matrix σ. $\square$

The Galois correspondence theorem is the fundamental result in Galois theory for polynomial equations as well as in differential Galois theory, and one can translate it easily by considering only closed subgroup for the Zariski topology. For any subgroup H of $\mathrm{Gal}(K/k)$, and any subfield L of K, denote K^H the set of elements of K fixed by H.

Theorem 4.3 (Galois correspondence)
Let K *be a Picard-Vessiot field with differential Galois group* G *over* k*.*

(i) There is a one-to-one correspondence between Zariski-closed subgroups $H \subset G$ and differential subfields F, $k \subset F \subset K$, given by

$$H \subset G \to K^H = \{a \in K, \sigma(a) = a \ \forall \sigma \in H\}$$

$$F \to \mathrm{Gal}(K/F) = \{\sigma \in G, \sigma(a) = a \ \forall a \in F\}$$

(ii) A subgroup $H < G$ is normal, if and only if K^H is invariant under G, in which case the canonical map $G \to$Gal(L^H/k) has kernel H.

(iii) A differential subfield F, $k \subset F \subset K$, is a Picard-Vessiot extension of k if and only if Gal(K/F) is a normal subgroup of G, in which case Gal$(F/k) \simeq G/$Gal(K/F).

The proof is slightly different than the one for polynomials, one can see [59]. The Galois group being a closed subgroup of $Gl_n(k_0)$ for a linear differential equation of order n, it also has a structure of Lie group. Hence, we can define its identity connected component $G0$. This will be the fundamental object to study integrability.

Proposition 4.3
K^{G_0} is the algebraic closure of k in K, and is a finite Picard-Vessiot extension with Galois group Gal(G/G^0).

Now that we have the main tools to deal with linear differential equations, we need a good notion of integrability. For a differential equation to be integrable, we want the solutions to be compositions of integrals and exponentials of the coefficients. This gave birth to the notion of Liouvillian extension, by analogy with the Galoisian one in the theory of polynomial equations.

Definition 4.6 (Liouvillian extension)
An differential extension L of k is said to be Liouvillian if its field of constant is k_0 and there exists a tower of field $k = L_0 \subset L_1 \subset \cdots \subset L_n = L$, such that $L_i = L_{i-1}(a_i)$ with either

(i) $t_i' \in L_{i-1}$ (integral),

(ii) $t_i'/t_i \in L_{i-1}$ (exponential)

(iii) t_i is algebraic over L_{i-1}.

We can think of the integrability of a linear differential equation as the fact that its Picard-Vessiot field is Liouvillian. Classically, a group G is said to be solvable if there exists a tower of group $\{e\} < G_1 < \cdots < G_n = G$, such that each quotient G_i/G_{i-1} is Abelian. We can now enunciate the main result of this theory:

Theorem 4.4
G^0 is solvable $\Leftrightarrow$ K is a Liouvillian extension of k $\Leftrightarrow$ K is contained in a Liouvillian extension of k.

The study of a linear differential equation, or more precisely, the question of whether or not we can calculate its solution, can be tackle via the group theory. We make the remark that in particular, if G^0 is Abelian, the equation is integrable. The local differential Galois group in x is the Galois group over the base field $k = \mathbb{C}_x(\{z\})$: the field of meromorphic functions in a neighborhood of x.

Monodromy, Galois group, and obstruction to integrability of Hamiltonian systems

In this thesis we are interested in Hamiltonian dynamics coming from optimal control problem, and as such, our differential equations are not linear. In this short subsection we briefly describe how the previous can be applied to Hamiltonian systems. The process actually goes back to Poincaré: On can select a particular integral curve of H and consider the variational equation along that particular curve, and then use the tools of the linear theory. Poincaré proved a result when the integral curve was a periodic solution, and that is one of the reasons why the study of periodic solutions in the three body problem is so important. Later, Ziglin in the 80's, and recently Moralès and Ramis, showed that idea was useful in general. We mainly work in the complex setting, so that the integral curves we will work with are actually Riemann surfaces. A local (but most of the time, global) parametrization on those curves will be the complex time t. We recall that an algebraic group G is said to be virtually Abelian if its connected component containing the identity (G^0) is an Abelian subgroup of G. The following celebrated theorem will be of great use:

Theorem 4.5 (Moralès-Ramis [41])
Let H be an analytic Hamiltonian on a complex analytic symplectic manifold and Γ be a non constant solution. If H is integrable in the Liouville sense with meromorphic first integrals, then the first order variational equation along Γ has a virtually Abelian Galois group over the base field of meromorphic functions on Γ.

The main idea behind this theorem is that if H is Liouville integrable, then so are the linearized equations near a non constant solution Γ. More precisely, thanks to Ziglin's Lemma below, the first integrals of H can be transformed in such a way that their first non trivial term in their series expansion near Γ are functionally independent.

Lemma 4.1 (Ziglin's Lemma)
Let $\Phi_1, \ldots, \Phi_r \in k(x_1, \ldots, x_n)$ be functionally independent functions. We consider $\Phi_1^0, \ldots, \Phi_r^0$ the lowest degree homogeneous term for some fixed positive weight homogeneity in $x_1, \ldots, x_n$. Assume $\Phi_1^0, \ldots, \Phi_{r-1}^0$ are functionally independent. Then there exists a polynomial Ψ such that the lowest degree homogeneous term Ψ^0 of $\Psi(\Phi_1, \ldots, \Phi_r)$ is such that $\Phi_1^0, \ldots, \Phi_{r-1}^0, \Psi^0$ are functionally independent.

Applying this Lemma recursively, one can prove that if a Hamiltonian system admits a set of commuting, functionally independent meromorphic first integrals on a neighborhood of a curve, then their first order terms, after possibly polynomial combinations of them, are also commuting, functionally independent meromorphic first integrals of the linearized system along the curve. Moralès-Ramis [41] precisely proved that symplectic linear differential systems having such first integrals have a Galois group whose identity component is Abelian. This result can be expected knowing that the Galois group leaves invariant every first integral, so the more first integrals, the smaller the Galois group.

Before applying this theory, let us linger a bit on the notion monodromy. The monodromy group of a linear differential equation is define the analytic continuations of the solutions along the loops around their poles. It is thus a notion linked with branching, or multi-valuation. More precisely, consider a differential system $Y' = AY$, $A \in M_n(\mathbb{C}(x))$. We note $S = \mathbb{P}^1 \backslash \{\text{singularities of } A\}$ (recall $\mathbb{P}^1$ is the Alexandrov compactification of $\mathbb{C}$). Let us consider a point $z_0 \in S$ and a closed oriented curve $\gamma \subset S$, with $x_0 \in \gamma$. There exists a basis of solutions Z on a neighborhood of x_0, holomorphic in z. We now use analytic continuation along the loop γ to extend this basis of solutions. However, it cannot *a priori* be extended to a whole neighborhood of γ, because after one loop, the basis of solutions Z_γ at x_0 could be different. This defines a matrix $D_\gamma \in GL_n(\mathbb{C})$ such that $Z_\gamma = ZD_\gamma$ and thus a homomorphism

$$\text{Mon} : \pi_1(S, x_0) \to GL_n(\mathbb{C}), \quad \text{Mon}(\gamma) = D_\gamma.$$

This homomorphism carries the group structure of $\pi_1(S, x_0)$, and thus its image is also a group.

Definition 4.7
The image of the application Mon is called the monodromy group.

Note that the monodromy group depends on the choice of Z, so it is only determined up to conjugation. Since analytic continuation preserves analytic relations, the monodromy group is a subset of the differential Galois group over the base field of meromorphic functions on S; in particular, it is included in the differential Galois group over the base field of rational functions. It is not a Lie group because it is not closed in general. To explore the relation between the monodromy group and the Galois group we need the notion of Fuchsian equations.

Definition 4.8 (Regular-singular point)
A singular point t^* of a differential equation with meromorphic coefficients a_i, $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$, is said to be regular if for all k, the kth coefficient a_k has a pole of order $\leqslant n-k$, ie: $(t-t^*)^{n-k}a_k(t)$ is holomorphic at t^*.

An differential equation is called Fuchsian if all its singularities are regular-singular points. To define the Fuchsian condition on multidimensional equations, one can just write them in scalar form via the cyclic vector method described in the following section. The next theorem will be of use to extend our theorem to a larger class of functions.

Theorem 4.6 (Schlesinger density theorem [53])
Let $(E): \ Y' = AY$ be a Fuchsian differential linear equation with coefficients in $\mathbb{C}(x)$ and let Π be its monodromy group. Then Π is dense for the Zariski topology in the Galois group of the Picard-Vessiot extension of (E) over the base field of rational functions: $\overline{\Pi} = Gal(A)$.

With Ziglin's theory, (see [60] for instance), one study the monodromy on the linearized equation along a integral curve in the complex domain, which supposedly contains more obstruction (here given by non-commutativity in the monodromy group) than in the real one. Galois differential theory of Moralès and Ramis is only relevant on a algebraically closed field. Most of the time, one consider real systems and complexify the Hamiltonian vector field X_H on a complexification of the state space. Then, the non-integrability result says that there is not enough analytic (or rational or even meromorphic) first integrals which has a complex extension. Nevertheless, let us enunciate a result about real non-integrability due to Audin, in [7]. A Hamiltonian system with n degree of freedom is said to be real integrable if there exist n real analytic (meromorphic) first integrals in involution.

Theorem 4.7
Let x be a real point of a complex trajectory Γ of the complexified Hamiltonian vector field X_H. If the local Galois group in x of the variational equation along Γ is not virtually Abelian , then the original real system is not integrable.

4.2 Integrability and its obstructions in Hamiltonian systems coming from optimal control

This section is a more detailed version of the paper [45], we prove the non integrability of the minimum time Kepler problem, and compare it with what occurs when minimizing other criteria, as the energy. The Kepler problem of example 4.2

$$\ddot{q} + \frac{q}{\|q\|^3} = 0, \quad q \in \mathbb{R}^2 \backslash \{0\}. \tag{4.2.1}$$

is a classical reduction of the two-body problem [5]. Here, we think of q as the position of a spacecraft, and of the attraction as the action of the Earth. We are interested in controlling the transfer of the spacecraft from one Keplerian orbit towards another, in the plane. Denoting $v = \dot{q}$ the velocity, and the adjoint variables of q and v by p_q and p_v, the minimum time dynamics is a Hamiltonian system with

$$H(q, v, p_q, p_v) = p_q.v - \frac{p_v.q}{\|q\|^3} + \|p_v\|, \tag{4.2.2}$$

as is explained in section 4.2.1. Prior studies of this problem can be found in [20, 23]. The controlled Kepler problem can be embedded in the two parameter family obtained when considering the control of the circular restricted three-body problem of example 4.3:

$$\ddot{q} + \nabla_q \Omega_\mu(t, q) = \varepsilon u, \tag{4.2.3}$$

where

$$\begin{aligned} \Omega_\mu(t,q) \quad = \quad & -\frac{1-\mu}{\sqrt{(q_1 + \mu\cos t)^2 + (q_2 + \mu \sin t)^2}} \\ & -\frac{\mu}{\sqrt{(q_1 - (1-\mu)\cos t)^2 + (q_2 - (1-\mu)\sin t)^2}} \end{aligned}$$

is the potential parameterized by the ratio of masses, $\mu \in [0, 1/2]$, and where $u \in \mathbb{R}^2$ is the control, whose amplitude is modulated by the second parameter,

$\varepsilon \geqslant 0$. Alternatively to time minimization, minimization of the $\mathcal{L}^2$ norm of the control can be considered,

$$\int_0^{t_f} u^2(t)\,\mathrm{d}t \to \min .$$

This is the so-called energy cost. In the uncontrolled model ($\varepsilon = 0$), it is well known that the Kepler case ($\mu = 0$) is integrable and geodesic (there exists a Riemannian metric such that Keplerian curves are geodesics of this metric [44, 46]) while there are obstructions to integrability for positive μ. In the controlled case ($\varepsilon > 0$), the Kepler problem for the energy cost has been shown to be integrable (and geodesic) when suitably averaged (see [21] for a survey). The aim of this chapter is to study the integrability properties of the Kepler problem for time minimization, and to compare with the energy optimization.

The pioneering work of Ziglin in the 80's [60], followed by the modern formulation of differential Galois theory in the late 90's by Moralès, Ramis and Simó [41, 42], have led to a very diverse literature on the integrability of Hamiltonian systems. According to Pontrjagin's Maximum principle, one can turn general optimization problems with dynamical constraints into Hamiltonian systems, which are generally not everywhere differentiable. Optimal control theory thus provides an abundant class of dynamical systems for which integrability is a central question. Yet, differential Galois theory has not so often been applied in this context (see, *e.g.*, [14]), in part because of the difficulty brought by the singularities. Notwithstanding theses singularities (vanishing of the adjoint variable p_v, here), we show how to apply these ideas to the system (4.2.2).

4.2.1 Setting

The minimum time controlled Kepler problem

We first recall some classical facts on optimal control. We refer for example to the book of Agrachev and Sachkov [4] for more details. Let M be an n-dimensional smooth manifold and U an arbitrary subset of $\mathbb{R}^m$ (typically a submanifold with boundary). A controlled dynamical system is a smooth family of vector fields

$$f : M \times U \to TM$$

parameterized by the control values. Admissible controls are measurable functions valued in the subset U. A preliminary question is the following: Is some final state x_f accessible from some initial state x_0, *i.e.* does the system

$$\dot{x}(t) = f(x(t), u(t)), \quad u(t) \in U,$$

$$x(0) = x_0, \quad x(t_f) = x_f,$$

have a solution for some admissible control? The system is said to be controllable if the answer is positive for all possible initial and final states $x_0, x_f \in M$. The controlled Kepler problem, associated with (4.2.1), is

$$\ddot{q} + \frac{q}{\|q\|^3} = u, \quad q \in \mathbb{R}^2 \backslash \{0\}, \quad u_1^2 + u_2^2 \leqslant 1,$$

$$(q(0), \dot{q}(0)) = (q_0, v_0), \quad (q(t_f), \dot{q}(t_f)) = (q_f, v_f),$$

where q is the position vector of a spacecraft and where the control u is the thrust of the engine. The thrust is obviously bounded; here we assume that it is valued in the Euclidean unit ball. (Note that, with respect to (4.2.3), we have chosen $\varepsilon = 1$; as will be clear from Section 4.2.2, this does not restrict the generality of the analysis.)

Proposition 4.4 ([23])
The Kepler problem is controllable.

This is a consequence of two facts: The Lie algebra generated by the drift and the vector field supporting the control generate the whole tangent space at each point (which entails some local controllability), and the uncontrolled flow (or *drift*) of the Kepler problem is recurrent, then one apply theorem 1.1 to conclude. Under some additional convexity and compactness assumptions, one is then able to retrieve existence of optimal controls.

We now deal with such optimal controls. We restrict ourselves to integral cost functions, that is to problems of the form

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ x(0) = x_0, \quad x(t_f) = x_f, \\ \int_0^{t_f} L(x(t), u(t)) \, \mathrm{dt} \to \min \end{cases} \tag{4.2.4}$$

where the final time t_f can be fixed or not, and $L : M \times U \to \mathbb{R}$ is a smooth function. In the early 60's, Pontrjagin and his coauthors realized that necessary conditions for optimality could be stated in Hamiltonian terms. By T^*M we denote the cotangent bundle of the manifold M.

Definition 4.9
The associated pseudo-Hamiltonian is

$$H : T^*M \times \mathbb{R} \times U \to \mathbb{R}, \quad (x, p, p^0, u) \mapsto \langle p, f(x,u) \rangle + p^0 L(x,u).$$

The following fundamental result is Pontrjagin Maximum Principle [51] (see [4] for a modern presentation).

Theorem 4.8
If (x,u) solves (4.2.4), there exists a Lipschitzian function $p(t) \in T^*_{x(t)}M$, $t \in [0,t_f]$, a constant $p^0 \leqslant 0$, $(p(t),p^0) \neq 0$, such that, almost everywhere,

(i) (x,p) is a solution of the Hamiltonian system associated with $H(\cdot,\cdot,u(t))$:

$$\dot{x} = \frac{\partial H}{\partial p}(x,p,u), \quad \dot{p} = -\frac{\partial H}{\partial x}(x,p,u),$$

(ii) $H(x(t),p(t),u(t)) = \max_{v\in U} H(x(t),p(t),v)$.

Such curves (x,p) are called extremals. As a consequence of the maximization condition, the pseudo-Hamiltonian evaluated along an extremal is constant. Moreover, if the final time is free then this constant is zero.

This powerful result has some downsides. The Hamiltonian is defined on the cotangent bundle of the original phase space, and thus the dimension is doubled. Besides, the maximization condition, which "eliminates the control" and allows to obtain a truly Hamiltonian system in (x,p) only, might generate singularities (that is non-differentiability points of the maximized Hamiltonian which is in general only Lipschitz continuous as a function of time when evaluated along an extremal). The above theorem applies to time minimization with $L \equiv 1$ (and free final time). In this case, the non-positive constant p^0 is only related to the level of the Hamiltonian, and we will not mention it in the sequel as we will not discuss the implications of having *normal* ($p^0 \neq 0$) or *abnormal* ($p^0 = 0$) extremals.

The minimum time Kepler problem can be stated according to

$$\begin{cases} \ddot{q} + \frac{q}{\|q\|^3} = u, \quad \|u\| \leqslant 1, \\ (q(0),\dot{q}(0)) = (q_0,v_0), \quad (q(t_f),\dot{q}(t_f)) = (q_f,v_f), \\ t_f \to \min, \end{cases} \tag{4.2.5}$$

where, as before, $q \in \mathbb{R}^2$ is the position vector and $u \in \mathbb{R}^2$ the control. It will be convenient to use the same notations as in the general problem (4.2.4) and let

$$q = (x_1,x_2), \quad \dot{q} = (x_3,x_4),$$

be the coordinates on the initial phase space $M = (\mathbb{R}^2\backslash\{0\}) \times \mathbb{R}^2$. According to Definition 4.9, the pseudo-Hamiltonian is then

$$H(x,p,u) = p_1x_3 + p_2x_4 - \frac{p_3x_1 + p_4x_2}{(x_1^2+x_2^2)^{3/2}} + p_3u_1 + p_4u_2. \tag{4.2.6}$$

According to Theorem 4.8, minimizing trajectories must be projections on M of integral curves of the Hamiltonian that has to be maximized over the unit disk. The maximized Hamiltonian is readily equal to

$$H(x,p) = p_1x_3 + p_2x_4 - \frac{p_3x_1 + p_4x_2}{(x_1^2+x_2^2)^{3/2}} + \sqrt{p_3^2+p_4^2}$$

on T^*M, while the control is given by

$$u = \frac{1}{\sqrt{p_3^2+p_4^2}}(p_3,p_4)$$

whenever p_3 and p_4 do not vanish simultaneously.

Main result

Let

$$\mathcal{M} = \{(x,p,r) \in \mathbb{C}^8 \times \mathbb{C}^2_*,\ r_1^2 = x_1^2+x_2^2,\ \ r_2^2 = p_3^2+p_4^2\}$$

be the Riemann surface of H. It is a complex symplectic manifold (with local Darboux coordinates (x,p) outside the singular hypersurface $r_1r_2 = 0$), over which H extends meromorphically, and even rationally, since

$$H(x,p,r) = p_1x_3 + p_2x_4 - \frac{p_3x_1+p_4x_2}{r_1^3} + r_2. \tag{4.2.7}$$

The Hamiltonian H has four degrees of freedom, hence (see [5]) the meromorphic Liouville integrability of H over $\mathcal{M}$ would mean that there would exist three independent first integrals, in addition to H itself, almost everywhere in $\mathcal{M}$. We will prove:

Theorem 4.9
The minimum time Kepler problem is not meromorphically Liouville integrable on $\mathcal{M}$.

It is well known that the classical Kepler problem is integrable, and even super integrable (since there are more first integrals than degrees of freedom, as a result of Kepler's first law and of the dynamical degeneracy of the Newtonian potential—see for instance [29]). On the opposite, the three-body problem is not as is known after the seminal work of Poincaré (for recent accounts on this topic see, *e.g.*, [25, 34, 49, 57]). Similarly, the above theorem asserts that lifting the Kepler problem to the cotangent bundle and introducing the singular control term r_2 breaks integrability.

This result prevents the existence of enough complex analytic (and even meromorphic) first integrals to ensure integrability over $\mathcal{M}$. Or course, it does not prevent the existence of an additional real first integral which would have a natural frontier asymptotic to the real domain and thus, would not extend to the complex plane.

Future work might be dedicated to investigate either or not Theorem 4.9 holds for real first integrals.

4.2.2 Proof of Theorem 4.9

The rest of the section is devoted to proving theorem 4.9. Our proof consists in studying the variational equation along some integral curve of (4.2.7). In order to carry out this computation, we choose a collision orbit, with the drawback that it requires some regularization. We also note that there exist effective tools to perform this kind of computations (see, *e.g.*, [26]).

A collision orbit

In order to find an explicit solution of 4.2.6, let us define the 4-dimensional symplectic submanifold

$$S = \{(x, p, r) \in \mathcal{M} \mid x_2 = x_4 = p_2 = p_4 = 0, r_1 = x_1, r_2 = -p_3\}.$$

As S is the phase space of the controlled Kepler problem on the line (collision orbit) parameterized by q_1, it is invariant. On the interior of S, (x_1, x_3, p_1, p_3) is a set of (Darboux) coordinates and, in restriction to S, the Hamiltonian reduces to

$$H(x, p) = p_1 x_3 - \frac{p_3}{x_1^2} - p_3,$$

so the Hamiltonian vector field on S is

$$\begin{cases} \dot{x}_1 &= x_3 \\ \dot{x}_3 &= -1 - \frac{1}{x_1^2} \\ \dot{p}_1 &= -\frac{2p_3}{x_1^3} \\ \dot{p}_3 &= -p_1. \end{cases}$$

In particular,

$$\begin{cases} \ddot{x}_1 &= -1 - \frac{1}{{x_1}^2} \\ \ddot{p}_3 - \frac{2p_3}{x_1^3} &= 0. \end{cases} \tag{4.2.8}$$

As is known since the work of Charlier and Saint Germain on the Kepler problem with a constant force (see [10]), the function

$$C = \frac{1}{2}x_3^2 + x_1 - \frac{1}{x_1}$$

is a first integral on S and $H_{|S}$ is integrable. Let us change time to $s = x_1(t)$ and denote by $' = \frac{d}{ds}$ the derivation with respect to this new time. It suffices to find an obstruction in this modified time, as explained at the end of the proof.

Using (4.2.8), we see that the variable p_3 satisfies the linear differential equation

$$2\left(C + \frac{1}{x_1} - x_1\right)p_3''(x_1) - \left(1 + \frac{1}{x_1^2}\right)p_3'(x_1) - \frac{2p_3(x_1)}{x_1^3} = 0,$$

which yields

$$p_3(x_1) = \frac{\sqrt{-Cx_1 + x_1^2 - 1}}{\sqrt{x_1}}\left(c_1\int\frac{x_1^{3/2}}{(-Cx_1 + x_1^2 - 1)^{3/2}}dx_1 + c_2\right)$$

for some constants of integration c_1 and c_2. Here the symbol $\int f(x_1)dx_1$ denotes some primitive of f with respect to the variable x_1. It suffices to find one particular integral curve along which the variational equation has a non virtually Abelian Galois group. To this end, we consider the simple—but rich enough—case $c_1 = 0$, $c_2 = 1$.

$$p_3(x_1) = \frac{\sqrt{-Cx_1 + x_1^2 - 1}}{\sqrt{x_1}}.$$

Using the expression of the first integral C and of the vector field, we deduce

$$x_3(x_1) = \sqrt{2}\frac{\sqrt{-Cx_1 + x_1^2 - 1}}{\sqrt{x_1}}, \quad p_1(x_1) = -\frac{1}{\sqrt{2}}\frac{x_1^2 + 1}{x_1^2}.$$

Choosing $C = 2i$ and some determination of the squares yields a particularly simple solution Γ drawn on $S \subset \mathcal{M}$,

$$\begin{cases} x_1 = x_1, \\ x_2 = 0, \\ x_3 = \sqrt{2}\frac{x_1 - i}{\sqrt{x_1}}, \\ x_4 = 0, \end{cases} \qquad \begin{cases} p_1 = -\frac{x_1^2+1}{\sqrt{2}x_1^2}, \\ p_2 = 0, \\ p_3 = \frac{x_1 - i}{\sqrt{x_1}}, \\ p_4 = 0. \end{cases} \tag{4.2.9}$$

Normal variational equation

In the initial time, the linearized equation along Γ is the Hamiltonian vector field associated with the Hamiltonian DH along Γ:

$$\dot{Z}(t) = A(t)Z(t), \quad A(t) = J\, D^2H(\Gamma(t)),$$

where J is the Poisson structure. In the coordinates $(x_1, ..., x_4, p_1, ..., p_4)$,

$$J = \begin{pmatrix} 0_4 & I_4 \\ -I_4 & 0_4 \end{pmatrix}.$$

We will keep on using time x_1, instead of the initial time t, writing

$$Z'(x_1(t)) = \frac{1}{x_3(t)} A(x_1(t))Z(x_1(t)).$$

Let us now reorder coordinates according to $(x_1, x_3, p_1, p_3, x_2, x_4, p_2, p_4)$. Since S is an invariant submanifold, the 8×8 matrix A has an upper triangular bloc structure

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

with

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}p_3} \\ -\frac{1}{\sqrt{2}p_3^2} & 0 & -\frac{1}{\sqrt{2}x_1^3 p_3} & 0 \\ 0 & \frac{1}{\sqrt{2}p_3} & 0 & 0 \\ -\frac{1}{\sqrt{2}x_1^3 p_3} & 0 & \frac{3}{\sqrt{2}x_1^4} & 0 \end{pmatrix}.$$

Moralès-Ramis Theorem gives necessary conditions for Liouville integrability in terms of the Galois group of this linear differential system over the base field of meromorphic functions *on* Γ. Looking at the expression (4.2.9) of Γ, we see that meromorphic functions on Γ are just meromorphic functions in $\sqrt{x_1} \in \mathbb{C}\backslash\{0, \pm\sqrt{i}\}$. The block A_3 corresponds to infinitesimal variations in the normal direction to S, which is the part where interesting phenomena might occur. As the Picard-Vessiot field is generated by all the components of the solutions, the Picard-Vessiot field K generated by the normal variational equation

$$(L): \quad X' = A_3 X, \quad X = (X_1, X_2, X_3, X_4)$$

is a subfield of the Picard-Vessiot field of the whole variational equation, and thus $\mathrm{Gal}(A) \supset \mathrm{Gal}(A_3)$. That $\mathrm{Gal}(A_3)$ is not virtually Abelian will thus imply that

Gal(A) itself is not virtually Abelian. In order to reduce the system to a one dimensional linear equation, we use the cyclic vector method on A_3: From (L) we get $X_1' = L_1(X_1, X_2, X_3, X_4)$, where L_1 is a linear form on $\mathbb{R}^4$, thus by derivation,

$$\begin{aligned} X_1'' =& L_1(X_1', X_2, X_3, X_4) + L_1(X_1, X_2', X_3, X_4) \\ &+ L_1(X_1, X_2, X_3', X_4) + L_1(X_1, X_2, X_3, X_4') \\ =& L_2(X_1, X_2, X_3, X_4). \end{aligned}$$

Iterating, we obtain

$$\begin{cases} X_1 = X_1, \\ X_1' = L_1(X_1, X_2, X_3, X_4), \\ X_1'' = L_2(X_1, X_2, X_3, X_4), \\ X_1^{(3)} = L_3(X_1, X_2, X_3, X_4), \\ X_1^{(4)} = L_4(X_1, X_2, X_3, X_4). \end{cases}$$

The L_i's are five linear forms on $\mathbb{R}^4$, so X_1 must satisfy some linear differential equation of order 4 that we compute to be

$$\begin{aligned} X_1^{(4)} + \frac{2(3i - 5x_1)}{x_1(i - x_1)} X_1^{(3)} + \frac{(-3x_1 + i)(-29x_1 + 23i)}{4(x_1 - i)^2 x_1^2} X_1'' \\ - \frac{(i - 3x_1)(7x_1 + i)}{4(x_1 - i)^2 x_1^3} X_1' + \frac{3x_1 + i}{4(x_1 - 1)^3 x_1^4} X_1 = 0. \end{aligned} \quad (4.2.10)$$

We find a solution of this equation of the form

$$X_1(x_1) = \frac{i - x_1}{\sqrt{x_1}} \left(c_1 + c_2 \int \sqrt{x_1}(1 + ix_1)^{-\frac{3}{2} - i\frac{\sqrt{3}}{2}} . {}_2F_1(\gamma(x_1)) dx_1 \right),$$

where ${}_2F_1$ is the Gauss hypergeometric function and

$$\gamma(x_1) = \left(\frac{5}{2} - i\frac{\sqrt{3}}{2}, \frac{1}{2} + i\frac{\sqrt{3}}{2}, 1 + i\sqrt{3}, 1 + ix_1 \right).$$

The Picard-Vessiot field K contains this solution and, as it is a differential field, it also contains

$$\sqrt{x_1}(1 + ix_1)^{-\frac{3}{2} - i\frac{\sqrt{3}}{2}} {}_2F_1(\gamma(x_1)).$$

Noting $\tilde{K}$ the differential field generated by this function, we have $\tilde{K} \subset K$. Now the Galois group of ${}_2F_1(\gamma(x_1))$ over $\mathbb{C}(x_1)$ is $SL_2(\mathbb{C})$ (see Kimura's table for the hypergeometric equations, [36]). By Galois correspondence, the Galois group of (4.2.10) over the rational functions in x_1 admits $SL_2(\mathbb{C})$ as a subgroup. The hypergeometric equation (4.2.10) is Fuchsian (all its singular points are regular), so thanks to Theorem 4.6, we know that its

Galois group over the field of rational functions is the closure of its monodromy group. Besides, the Galois group over meromorphic functions contains the monodromy group, and of course, is included in the Galois group over rational functions. Eventually, the Galois group of (4.2.10) over meromorphic functions in x_1 also contains $SL_2(\mathbb{C})$. Thus, adding the algebraic extension $\sqrt{x_1}$, the Galois group can be reduced to at most one subgroup of index 2: The only possibility is $SL_2(\mathbb{C})$ again. So the Galois group of K over the base field of meromorphic functions in $\sqrt{x_1} \in \mathbb{C}\backslash\{0, \pm\sqrt{i}\}$ contains $SL_2(\mathbb{C})$ and is not virtually Abelian. According to Morales-Ramis, this concludes the proof. □

4.3 Conclusion and perspectives

We conclude this chapter by enunciating some ideas and work in progress related to our main result. Theorem 4.9 is remarkable since it implies that the mere structure of the minimum time optimization perturbs the lifted Kepler problem in order to destroy integrability. The Hamiltonian lift of a vector field on a manifold M is defined by

$$H^{\text{lift}}(x,p) = \langle p, X(x)\rangle,\ (x,p) \in T^*M.$$

The following proposition is natural, and is a direct consequence of the work of Zung, in [8].

Proposition 4.5
The lift of an integrable system remains integrable.

Proof. Indeed, let H be a integrable Hamiltonian system on a symplectic manifold M and $f_1, \dots, f_n$ be n independent first integrals in involution. Then, the $\tilde{f}_i(x,p) := f_i(x)$, $(x,p) \in T^M$ are independent first integrals of $H^{\text{lift}} = \langle p, X_H(x)\rangle$, which commute, as well as the $F_i(x,p) = \langle p, X_{f_i}\rangle$, $i = 1, \dots, n$. Furthermore, the $\{\tilde{f}_i, F_i\} = 0$. Finally $\{F_i, F_j\} = \langle p, [X_{f_i}, X_{f_j}]\rangle = 0$ which conclude the proof. □

In optimal control problems, by the P.M.P., extremals are solution of the Hamiltonian given by the lifted dynamics, minus the criteria, maximized over all admissible controls. Thus, depending on the criteria to minimize, integrability could be preserved or destroyed, the maximization, potentially inducing singularities (and non linearity in the adjoint state p). For instance, if one chose to minimize a energy criteria on the controlled Kepler problem, ie, $\int_0^{t_f} \|u(t)\|^2 dt$, we end up with a system close to integrability: its average with respect to the fast variable (the angle on the ellipsis at each time) is integrable. Here, we see that time minimization creates obstructions to integrability but analysis of the Galois group in the minimal time control of coherence transfer for Ising chains let this time to integrability in [14]. Hence, influence of time minimization on integrable systems is still unclear: singularities are created, and they can preserve or destroy the existence of first integrals. However, we know from the proposition above that they are responsible for the lack of integrability here. It can be understand as followed: The Galois group contains the monodromy group, which is the group obtained by analytic continuation around the singularities of the differential equation. If this group is big enough, integrability cannot happen, and that is a consequence of the optimization procedure (namely, the P.M.P.). Since the lifted Kepler problem is integrable, via theorem 4.1, the cotangent bundle is foliated in stable torus. In a realistic model of orbit transfer, the thrust is very low compared to the gravitational force, thus we consider as in (4.2.3), the control in a small ball of radius ε. It becomes interesting to look at the problem from the angle of the perturbation theory of Kolmogorov, Arnold and Mother, see the annex of [5], or [30] for a modern review on KAM theory. To apply KAM theorem to the integrable Hamiltonian $H(q, v, p_q, p_v) = p_q.v - \frac{p_v.q}{\|q\|^3}$ with the small perturbation $\varepsilon\|p_v\|$, one has to answer the question is it possible to find action-angle coordinates for H making it convex. Then, the perseverance of KAM tori under this perturbation would imply the non geodesic convexity of the minimum time Kepler problem, as in the average energy case. This study may also be done for the (no average) minimum energy, but one should first prove its non-integrability.

Eventually, with the help of theorem 4.7, we should be able to prove non integrability in the class of real analytic first integrals, that do not even admit a extension to the complex domain. The issue is to choose wisely a level of the first integral C, on which the local Galois group of the real part of the trajectory is big enough (ie, non virtually Abelian).

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Résumé

Cette thèse contribue à l'étude en temps minimal des systèmes de contrôle affines. Les systèmes dépendant du contrôle de manière affine sont naturellement présents en physique et apparaissent dès qu'on s'intéresse aux systèmes mécaniques. Ils sont, pour autant, bien plus généraux. Dans ce manuscrit on traite les singularités de tels systèmes, en minimisant le temps final, celui où l'objectif est atteint. Une étude précise de leur flot extrémal est faite, d'abord pour les systèmes mécaniques, puis en général. Cela nous permet d'obtenir la régularité du flot, qui s'avère être lisse sur une stratification au voisinage du lieu singulier. Nous appliquons ensuite les résultats au problème du transfert d'orbite d'un engin spatial, et contrôlons le nombre singularités présentes au cours d'un transfert. Nous changeons ensuite de point de vue pour s'intéresser aux conditions d'optimalités des extrémales étudiées, et donnons un critère d'optimalité locale, calculable via un simple test numérique. Il est enfin question d'étudier ces singularités du point de vue de l'intégrabilité des systèmes Hamiltoniens : nous prouvons ainsi que le problème du transfert d'orbite à deux corps en temps minimal n'est pas intégrable au sens de Liouville.

Mots Clés

Contrôle optimal, contrôle géométrique, systèmes dynamiques, systèmes intégrables, systèmes Hamiltoniens, singularités, théorie de Galois différentielle, stratification.

Abstract

This thesis contribute to the optimal time study of control-affine systems. These problems arise naturally from physics, and contains, for instance, mechanical systems. We tackle the study of their singularities, while minimizing the final time, meaning the time on which the aim is reached. We give a precise study of the extremal flow, for mechanical systems, for starter, and then, in general. This leads to the knowledge of the flow regularity: it is smooth on a stratification around the singular set. We then apply those results to mechanical systems, and orbit transfer problems, with two and three bodies, giving an upper bound to the number of singularities occurring during a transfer. We then change our viewpoint to study the optimality of such extremal in general, and give an optimality criteria than can be easily checked numerically. In the last chapter we study the singularities of the controlled Kepler problem through another path: we prove a non-integrability theorem - in the Liouville sens - for the Hamiltonian system given by the minimum time orbit transfer (or rendez-vous) problem in the Kepler configuration.

Keywords

Optimal control, geometric control, dynamical systems, integrable systems, Hamiltonian systems, singularity theory, Galois differential theory, stratification.